Sorting monoids on Coxeter groups

A computer exploration with `Sage-Combinat`

Florent Hivert[1] Anne Schilling[2] **Nicolas M. Thiéry[2,3]**

[1]LITIS/LIFAR, Université Rouen, France

[2]University of California at Davis, USA

[3]Laboratoire de Mathématiques d'Orsay, Université Paris Sud, France

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arXiv:0711.1561v1 [math.RT]
arXiv:0804.3781v1 [math.RT]
arXiv:0912.2212v1 [math.CO]
+ research in progress

Résumé

Pour tout groupe de Coxeter finis W, nous définissons deux nouveaux objets: son ordre de coupures et son monoïde de Hecke double. L'ordre de coupures, construit au moyen d'une généralisation de la notion de bloc dans les matrices de permutations, est presque un treillis sur W.
La construction du monoïde de Hecke double s'appuie sur le modèle combinatoire usuel de la 0-algèbre de Hecke $H_0(W)$, i.e., pour le groupe symétrique, l'algèbre (ou le monoïde) engendré par les opérateurs de tri par bulles élémentaires. Les auteurs ont introduits précédemment l'algèbre de Hecke-groupe, construite comme l'algèbre engendrée conjointement par les opérateurs de tri et d'anti-tri, et décrit sa théorie des représentations.
Dans cet exposé, nous étudions le monoïde engendré par ces opérateurs, et nous expliquons comment la théorie des représentations et l'exploration informatique nous ont servi de guide pour mettre à jour une combinatoire riche, faisant intervenir les ordres usuels sur les groupes de Coxeter (Bruhat, permutohèdre gauche et droit) ainsi que l'ordre de coupure comme généralisation de la combinatoire des descentes.
L'exposé vise une large audience, s'appuyant sur de multiples d'exemples et sur des sessions de calculs typiques avec Sage.

Sage-Combinat (combinat.sagemath.org)

- 50+ research articles
- Sponsors: ANR, PEPS, NSF, Google Summer of Code
- Sage: 300 tickets / 100k lines integrated in Sage
- MuPAD: 115k lines of MuPAD, 15k lines of C++, 32k lines of tests, 600 pages of doc
- Nicolas Borie, Daniel Bump, Jason Bandlow, Adrien Boussicault, Vincent Delecroix, Paul-Olivier Dehaye, Tom Denton, Dan Drake, Teresa Gomez Diaz, Mike Hansen, Ralf Hemmecke, Florent Hivert, Brant Jones, Sébastien Labbé, Yann Laigle-Chapuy, Andrew Mathas, Gregg Musiker, Steven Pon, Franco Saliola, Anne Schilling, Mark Shimozono, Nicolas M. Thiéry, Justin Walker, Qiang Wang, Mike Zabrocki, ...

Bubble (anti) sort algorithm

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Underlying combinatorics: right permutohedron

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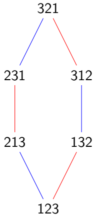

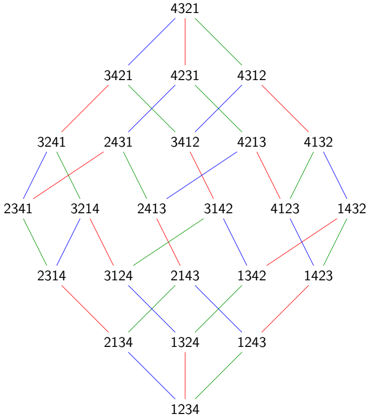

Bubble (anti) sort algorithm

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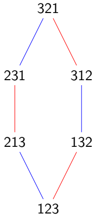
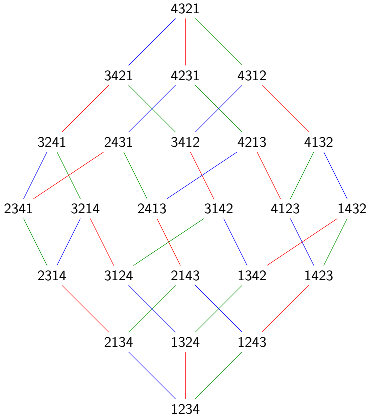
Underlying combinatorics: right permutohedron





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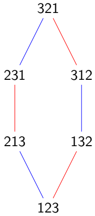

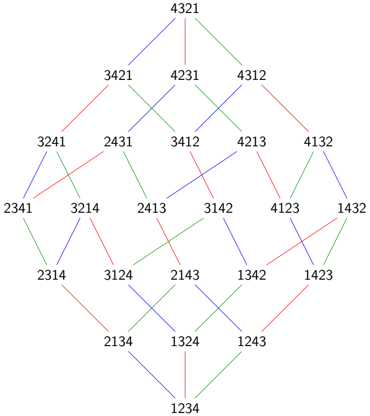

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Relations: $s_i^2, (s_1 s_2)^3 = 1, (s_2 s_3)^3 = 1, (s_1 s_3)^2 = 1$

Coxeter groups

Definition (Coxeter group W)

Generators : $s_1, s_2, \ldots$ (simple reflections)

Relations: $s_i^2 = 1$ and $\underbrace{s_i s_j \cdots}_{m_{i,j}} = \underbrace{s_j s_i \cdots}_{m_{i,j}}$, for $i \neq j$

- Reduced word
- Length

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Orders on words and on Coxeter group elements

Definition (Orders on words)

Let $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$:

- u **left factor** of v if $v = u_1 \cdots u_k \cdots$
- u **right factor** of v if $v = \cdots u_1 \cdots u_k$
- u **factor** of v if $v = \cdots u_1 \cdots u_k \cdots$
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Definition (Orders on Coxeter group elements)

- Right weak order
- Left weak order
- Left-right weak order
- Bruhat order

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Blocks of permutations

Definition (Block of a permutation w)

- Type A: sub-permutation matrix
- Type free: J, K such that $W_J w = w W_K$

- Example: $w := 36475812$
- Simple permutation: cf. [Albert, Atkinson 05] + dim 2 posets
- {blocks of w}: sub-lattice of the Boolean lattice

Definition (HST09: Cutting poset $(W, \sqsubset)$)

$u \sqsubset w$ if $u = w^J$ with J block

- (almost) lattice
- Möbius function: inclusion-exclusion along minimal blocks

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Hecke monoid

Definition (0-Hecke monoid $H_0(W)$ of a Coxeter group W)

Generators : $\langle \pi_1, \pi_2, \dots \rangle$ (simple reflections)
Relations: $\pi_i^2 = \pi_i$ and braid relations

Theorem

$|H_0(W)| = |W|$
+ lots of nice properties

Motivation: simple combinatorial model (bubble sort)
appears in Iwahori-Hecke algebras, Schur symmetric functions, Schubert, Kazhdan-Lusztig polynomials, and Macdonald, (affine) Stanley symmetric functions, mathematical physics, Schur-Weyl duality for quantum groups, representations of $GL(\mathbb{F}_q)$, ...

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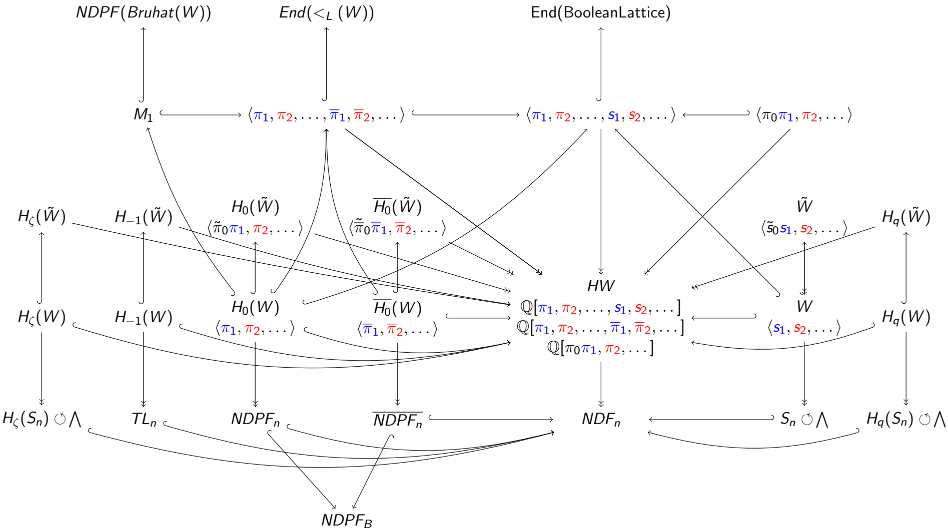
The Big Picture



The bi-Hecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots \rangle$

$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Theorem (HST08)

$M(W)$ admits $|W|$ simple / indecomposable projective modules

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w . \dim P_w$$

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Representation theory of algebras

Module: vector space V with a morphism $M \mapsto \mathrm{End}(V)$
Simple module: V contains no nontrivial submodule
Indecomposable module: V cannot be written as $V = V_1 \oplus V_2$
Projective module: $V \oplus \cdots = \mathbb{C}[M] \oplus \cdots \oplus \mathbb{C}[M]$

Theorem (See e.g. Curtis-Reiner)

Simple modules $\leftrightarrow$ indecomposable projective modules
Dimension formula, ...

Key role of idempotents:

- eV projective module: $V = eV \oplus (1-e)V$
- If $f = uev$ then fM is isomorphic to a submodule of eM

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Representation theory of monoids

Definition (J-(pre)order)

$x \leq_J y$ iff $x = uyv$, for some $u, v \in M$
$x, y \in M$ are in the same J-class if $x \leq_J y$ and $y \leq_J x$
A J-class is regular iff it contains an idempotent

Theorem (See e.g. Ganyushkin, Mazorchuk, Steinberg 07)

The regular J-classes determine the simple modules.

Definition (Schüztenberger)

Aperiodic monoid: no trivial subgroup

$\Longrightarrow$ Combinatorial Representation Theory

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Aperiodic monoid: no trivial subgroup

$\Longrightarrow$ Combinatorial Representation Theory

Representation theory of monoids

Definition (J-(pre)order)

$x \leq_J y$ iff $x = uyv$, for some $u, v \in M$
$x, y \in M$ are in the same J-class if $x \leq_J y$ and $y \leq_J x$
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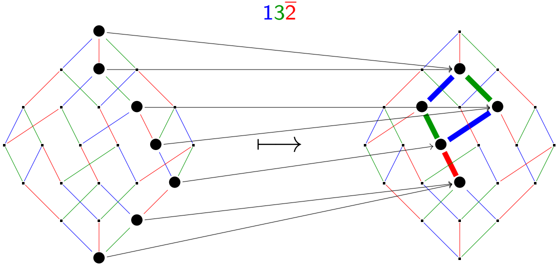

Lemma

For $f \in M(W)$ and $w \in W$: $\quad (s_i w).f = w.f \quad$ *or* $\quad s_i\,(w.f)$

Proof.

Exchange property / associativity □

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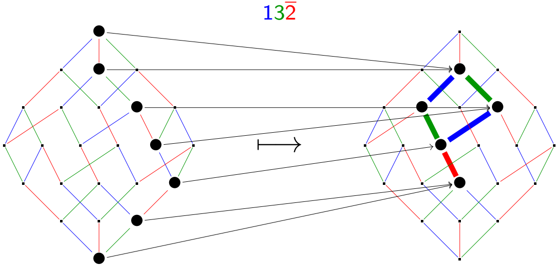

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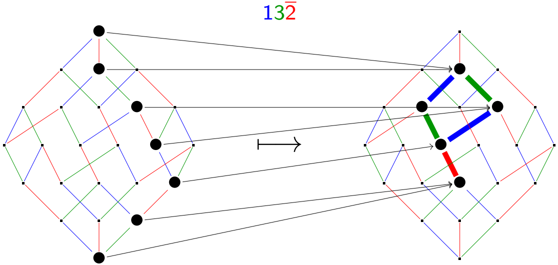

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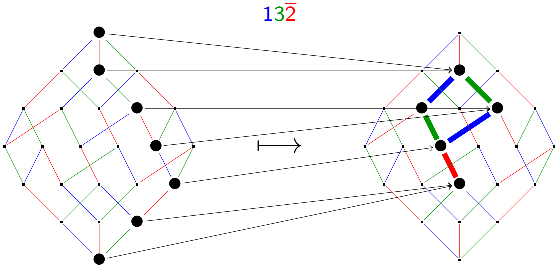

Corollary

- *If $w = uv$, then $(uv).f = u' \, (v.f)$, where $u' <_B u$*
- *Preservation of left order: $u \leq_L v \Longrightarrow u.f \leq_L v.f$*
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- *f in $M(W)$ is determined by its fibers and $f(1)$*

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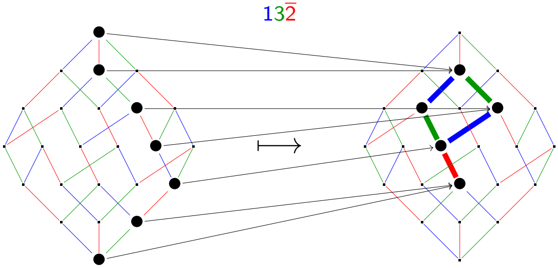

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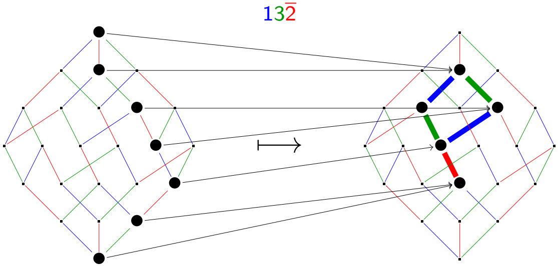

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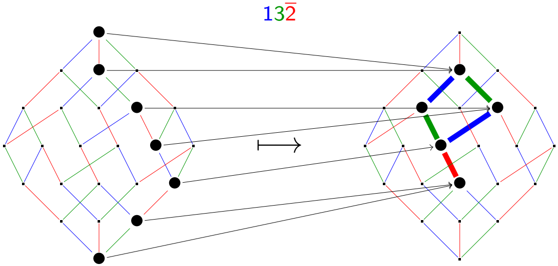

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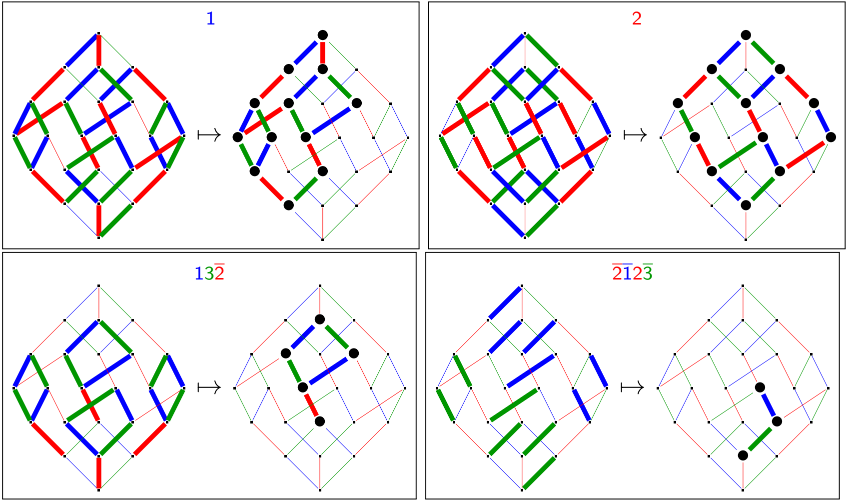
Some elements of the monoid



Representation theory of $M(W)$

Theorem (HST'08)

$M(W)$ admits $|W|$ simple modules

Sketch of proof.

- M acts transitively on intervals $[u, v]_L$
- The image set of an idempotent is an interval $[u, v]_L$
- $\exists!$ e_w idempotent with image set $[1, w]_L$, for any $w \in W$
- $(e_w)_{w \in W}$: transversal of the regular J-classes
  - $f = uev$ if and only if $\mathrm{im}(f)$ is a subinterval of $\mathrm{im}(e)$

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Problem

Dimension of simple and projective modules?

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Problem

Dimension of simple and projective modules?

The “Borel” submonoid M_1

Definition

Submonoid $M_1 := \{f \in M, f(1) = 1\}$

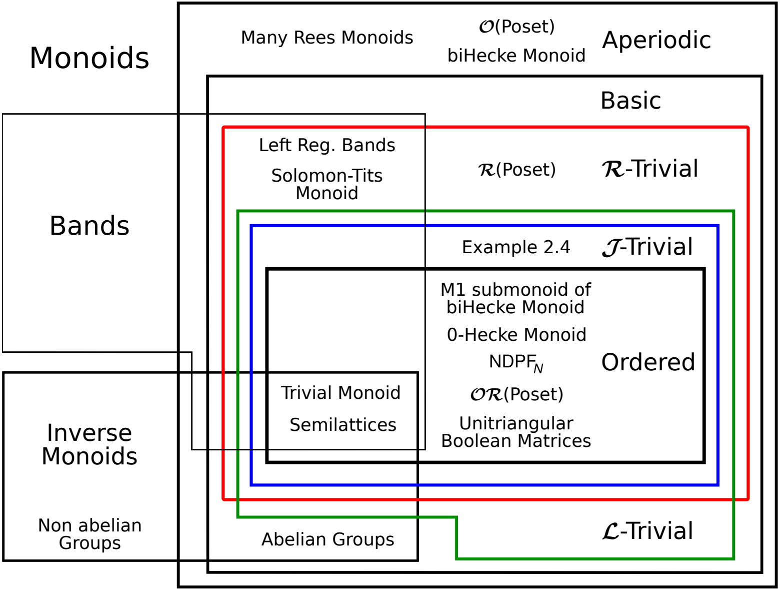
Properties (HST'09)

- *Weakly increasing and contracting on Bruhat* $\Longrightarrow$ *J-trivial*
- *Idempotents:* $(e_w)_{w \in W}$
- *Generated by* e_w *for* w *grassmanian (atom for* $(W, \vee_L)$*)*
- $|W|$ *simple modules of dimension* 1
- *Semi simple quotient: monoid algebra of* $(W, \vee_L)$
- *Conjugacy order among idempotents:* $<_L$
- $\dim P_w = |\{f \in M_1, f(w) = w...\}|$ *?*

Problem

Inducing these results to M*?*

Classes of monoids and representation theory



Representation theory of J-trivial monoids

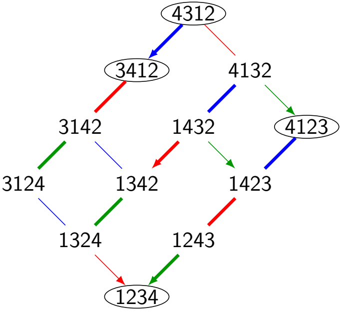
Theorem (HST'09)

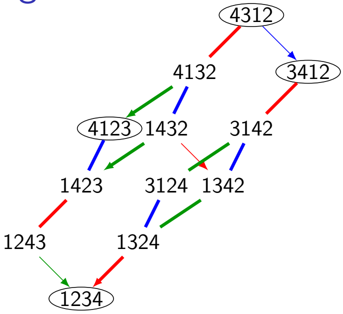
Combinatorial description of:

- *Simple modules*
- *Projective modules*
- *Cartan matrix*
- *Quiver*
- *q-Cartan matrix (in progress)*

in term of some statistic on M

Translation algebras





Definition (Translation algebra)

$T_w := \mathbb{Q}[\pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots]$ acting on $\mathbb{Q}.[1, w]_R$

- Blocks: $J = \{\}, \{1, 2\}, \{3\}, \{1, 2, 3\}$ $\implies$ Submodules P_J
- T_w: max. algebra stabilizing all P_J $\implies$ Repr. theory
- T_w quotient of $\mathbb{Q}[M(W)]$; top: simple module S_w of M
- Dimension: inclusion-exclusion along the cutting poset
- Generating series calculation?

Summary

- **Bubble sort** related monoid and algebras
- Typical question: **cardinality** ?
- Approach: **representation theory** + **computer exploration**

- Leads to interesting combinatorics:
  various **partial orders** on Coxeter groups
- Combinatorial representation theory of monoids:
  how to **eliminate linear algebra**?
- Effective algorithms and combinatorial results

Work in progress

- Radical filtration = length of the paths, for some particular combinatorial J-trivial monoids
- generalization to R-trivial and aperiodic monoids (collaboration with Denton and Berg, Bergeron, Saliola)
- Fast implementation is Sage (interface with `Semigroupe`, ...)
- Simple permutations and cutting poset