Sorting monoids and algebras on Coxeter groups

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New Orleans, March 20th of 2009

arXiv:0711.1561v1 [math.RT]
arXiv:0804.3781v1 [math.RT]
+ research in progress ...

Bubble (anti) sort algorithm

1234

Bubble (anti) sort algorithm

1234

Bubble (anti) sort algorithm

1243

Bubble (anti) sort algorithm

1423

Bubble (anti) sort algorithm

4123

Bubble (anti) sort algorithm

4123

Bubble (anti) sort algorithm

4132

Bubble (anti) sort algorithm

4312

Bubble (anti) sort algorithm

4312

Bubble (anti) sort algorithm

4321

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Underlying combinatorics: right permutohedron

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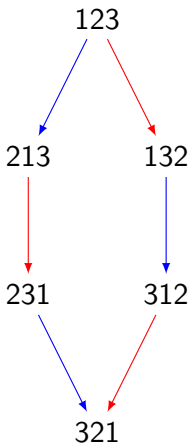
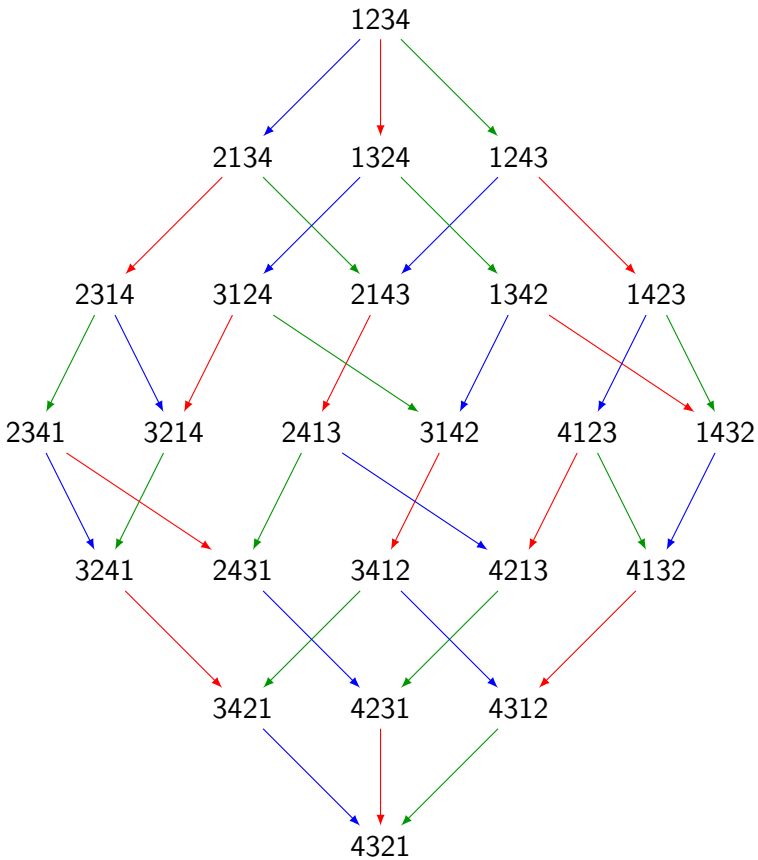
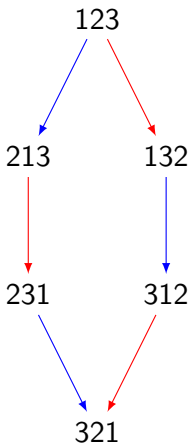


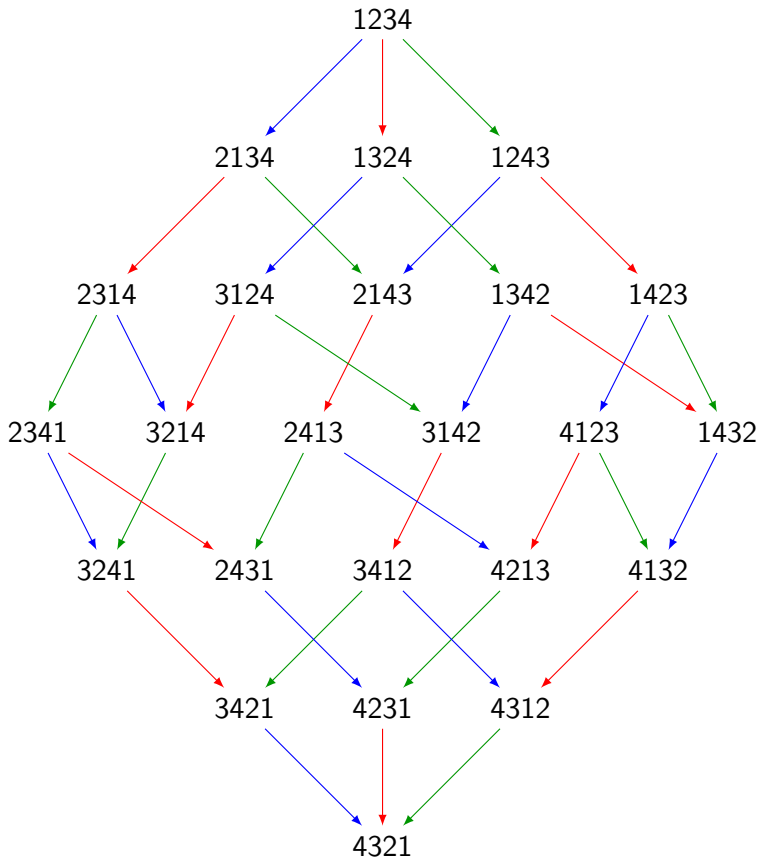


Bubble (anti) sort algorithm

4321

Underlying combinatorics: right permutohedron





Elementary transpositions: $s_1, s_2, s_3, \ldots$

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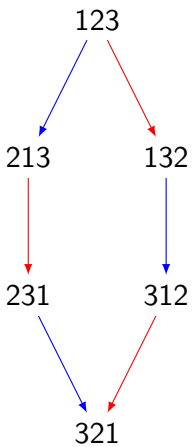
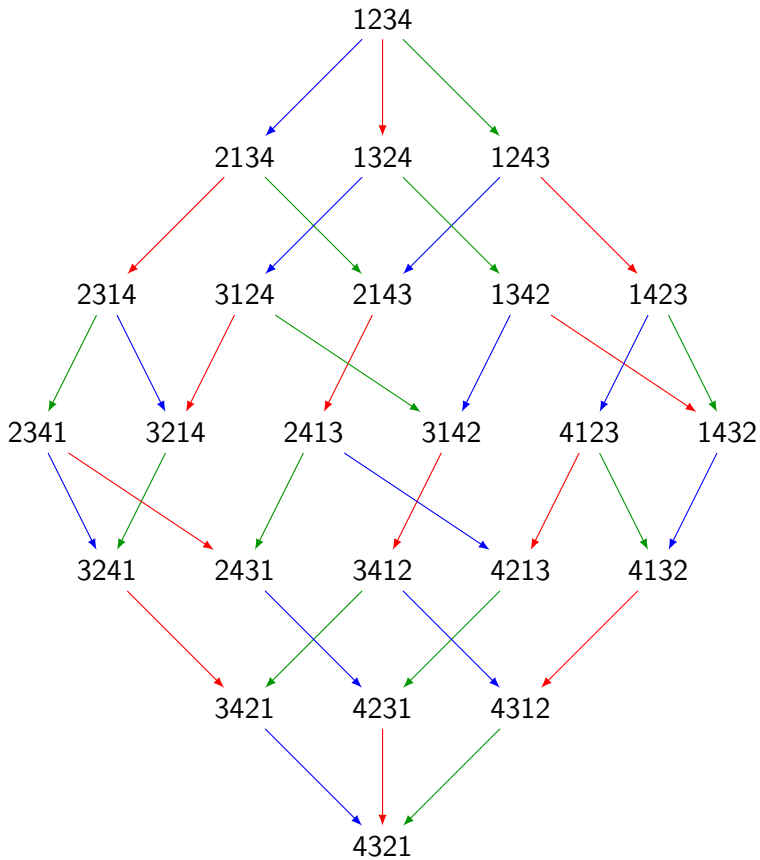


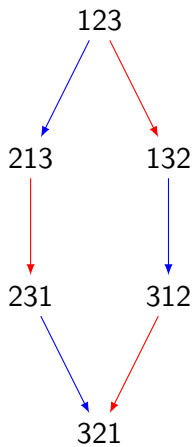


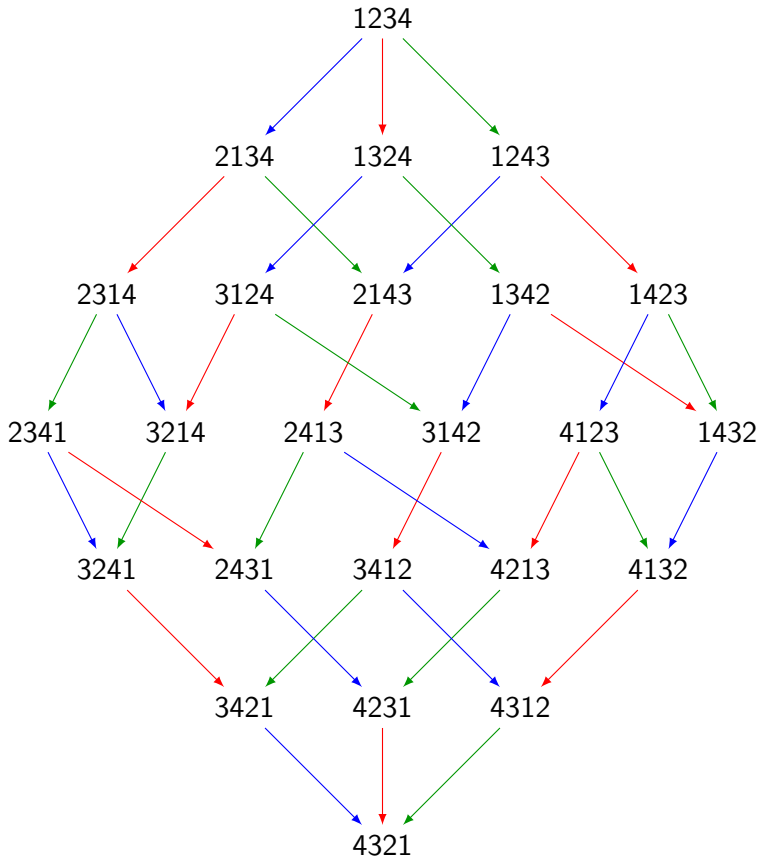
Elementary transpositions: $s_1, s_2, s_3, \ldots$

Elementary bubble antisort operators: $\pi_1, \pi_2, \pi_3, \ldots$

Cyclic bubble sort

$\pi_1, \pi_2, \pi_3, \ldots$ antisort 12345 into 54321 There is no return!





Proposition (HT'08)

$\pi_0, \pi_1, \pi_2, \pi_3, \ldots$ *act transitively on permutations*

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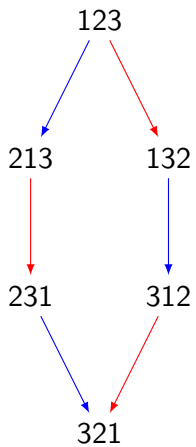
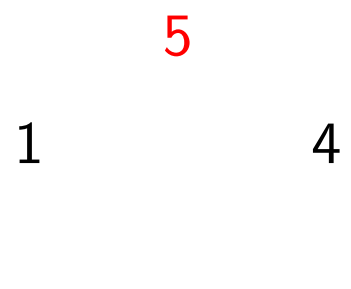
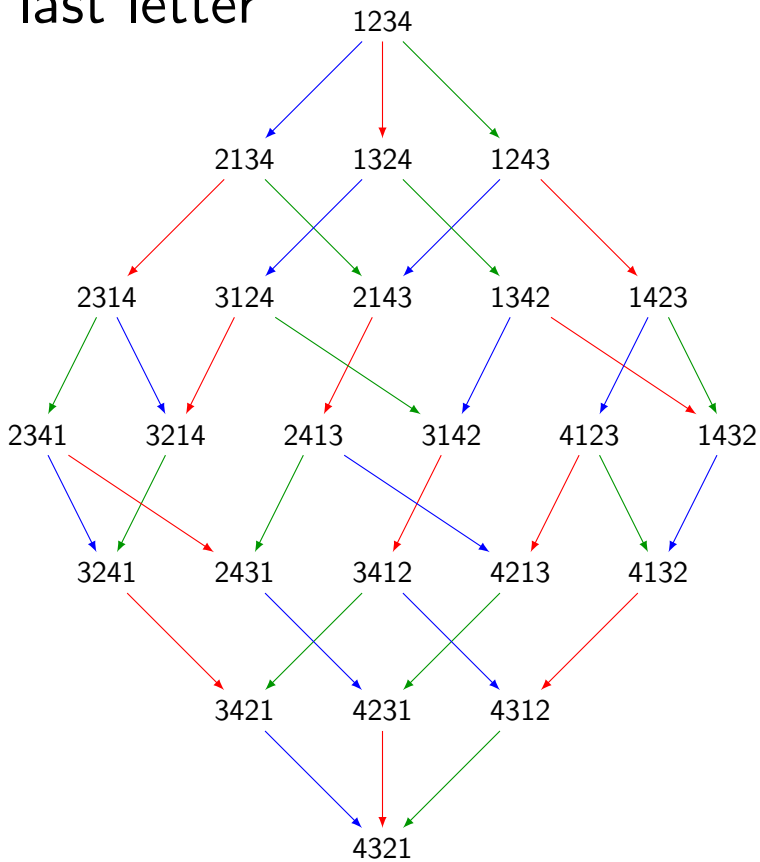
New operator π_0: acts between first and last letter







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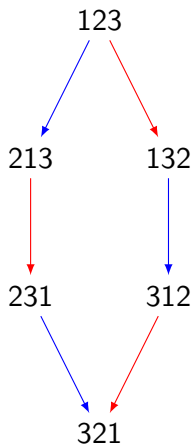

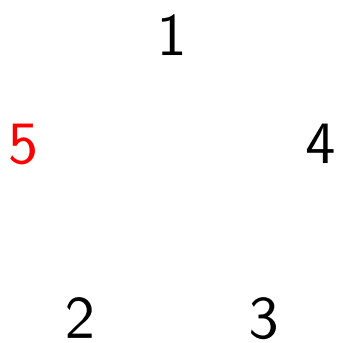

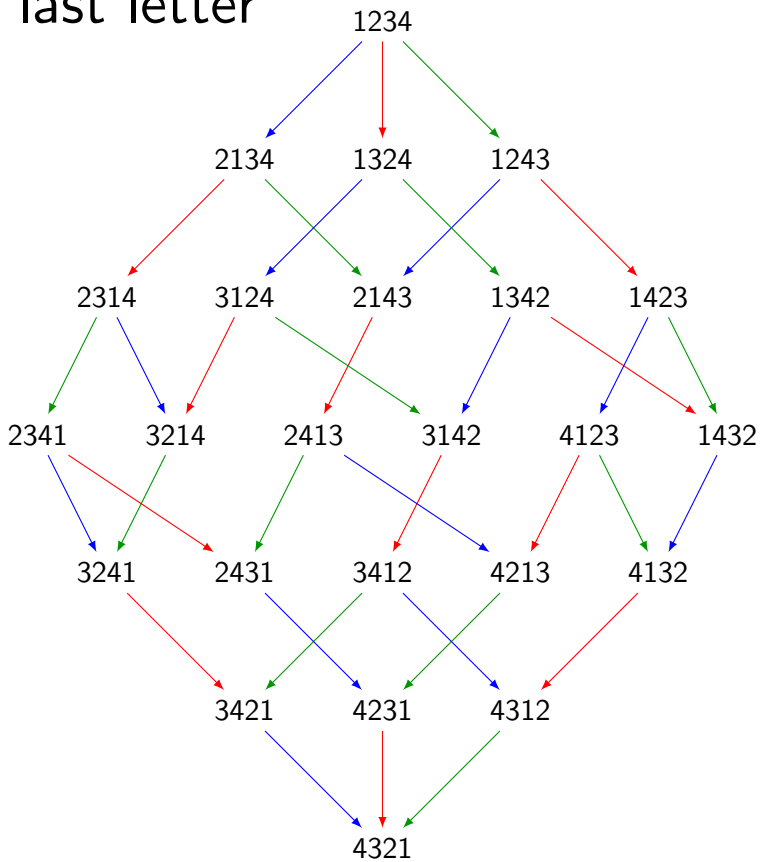

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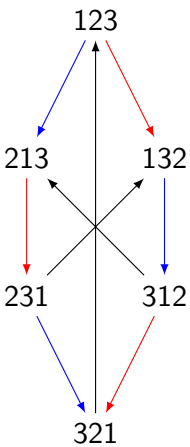

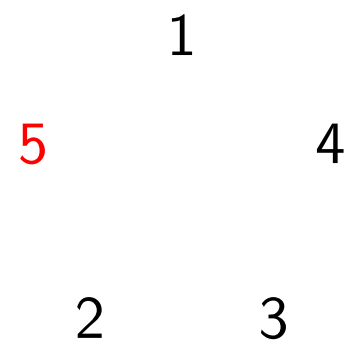

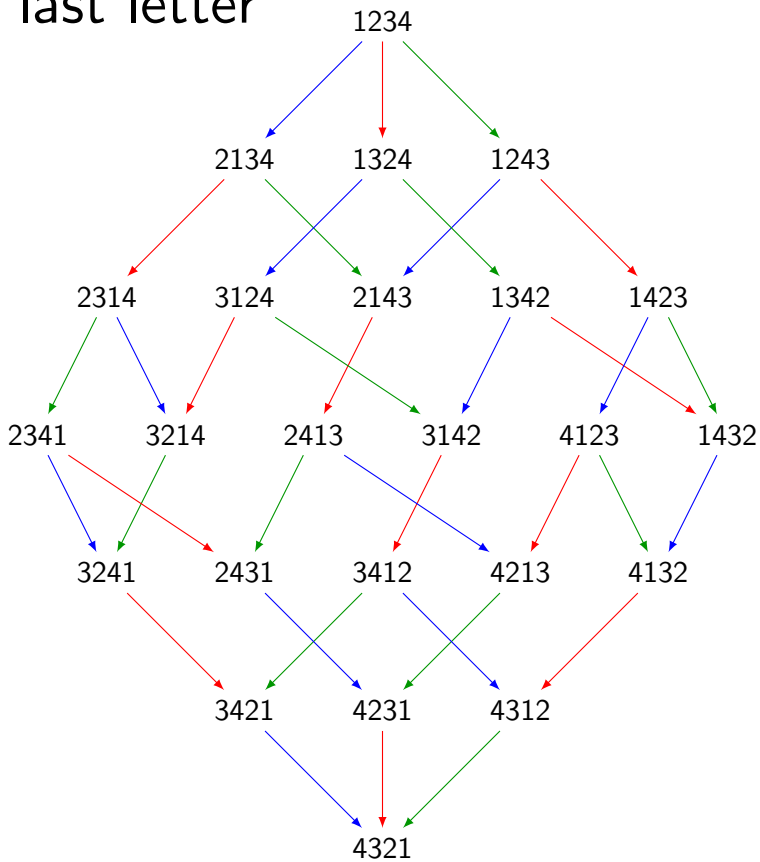

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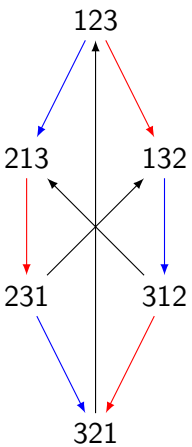

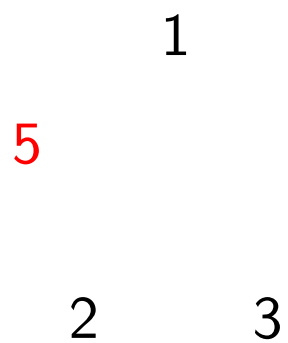

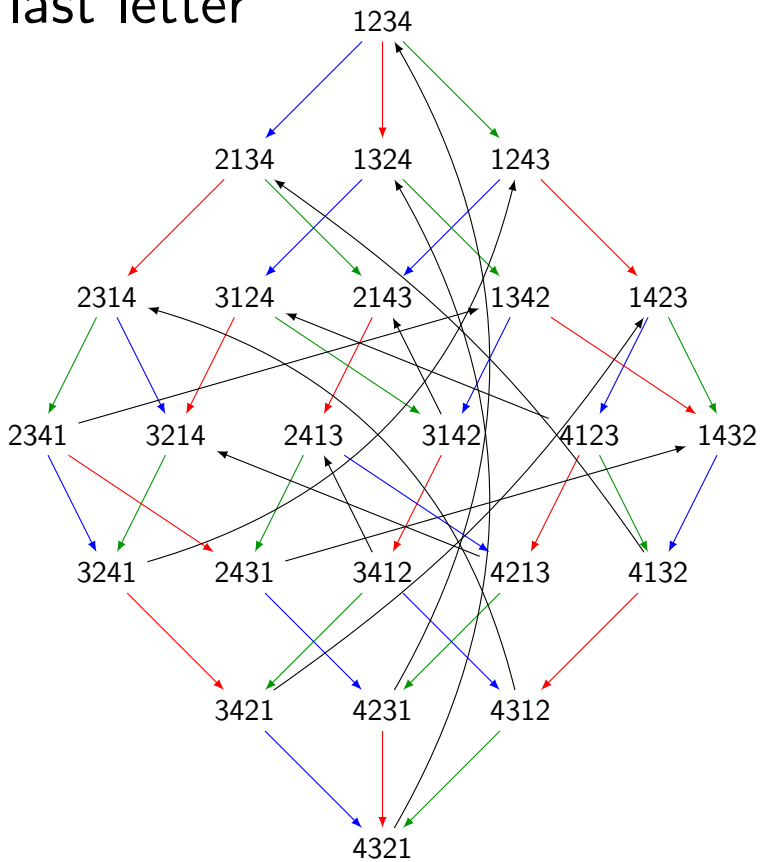

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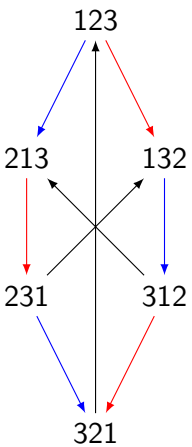

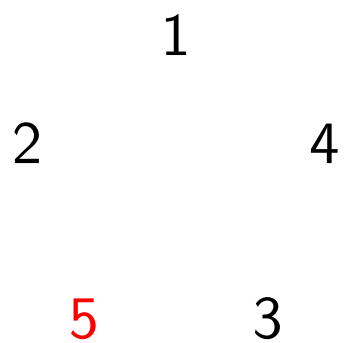

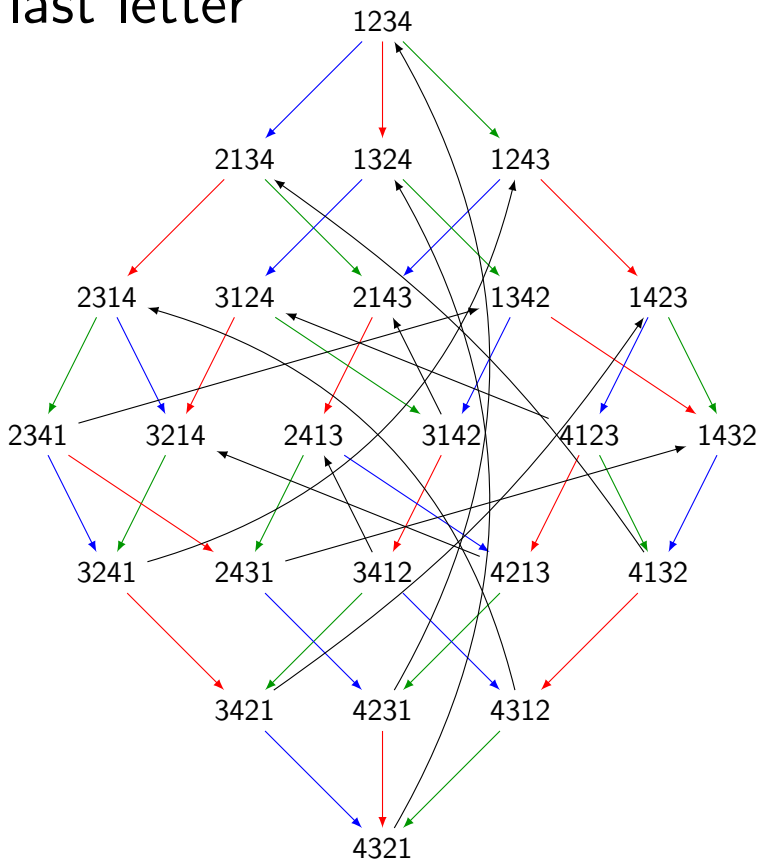

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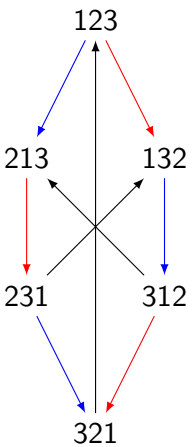

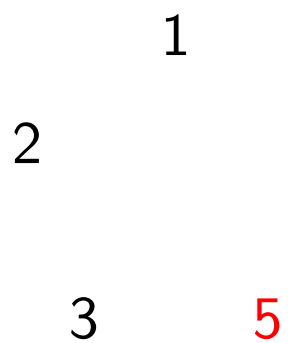

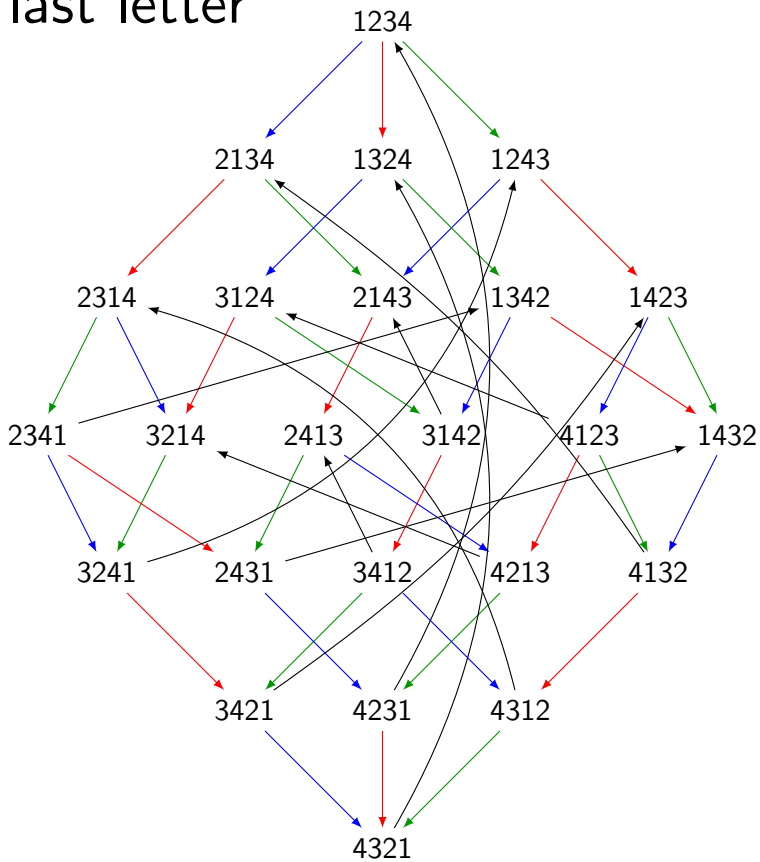

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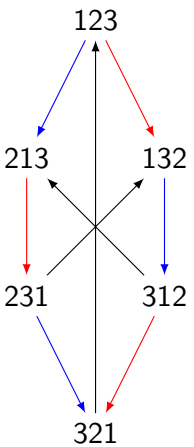

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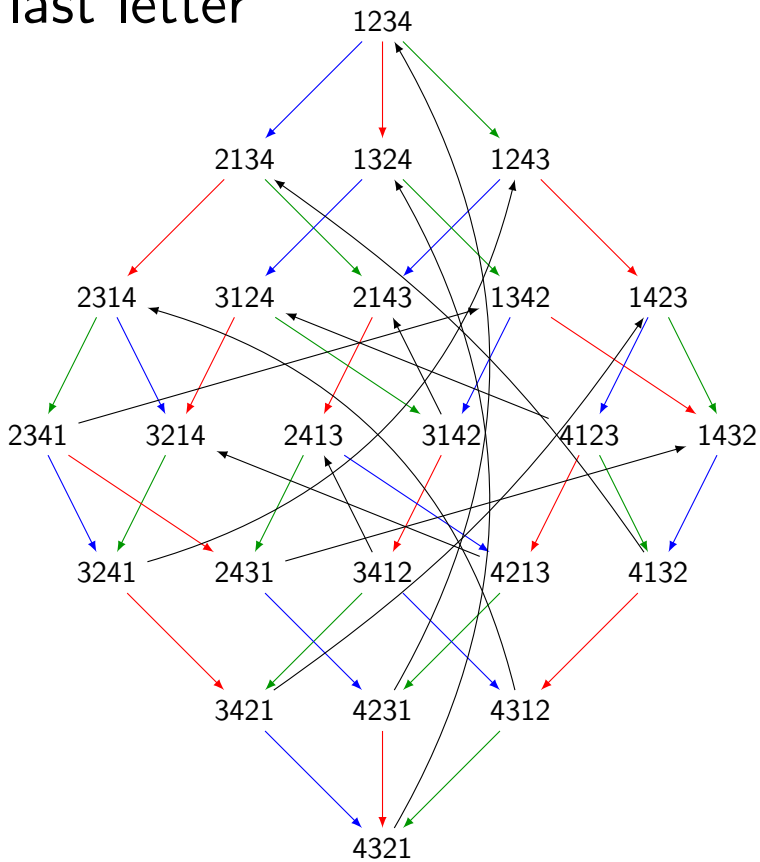

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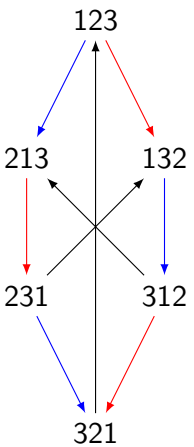

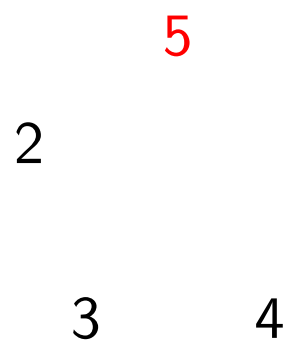

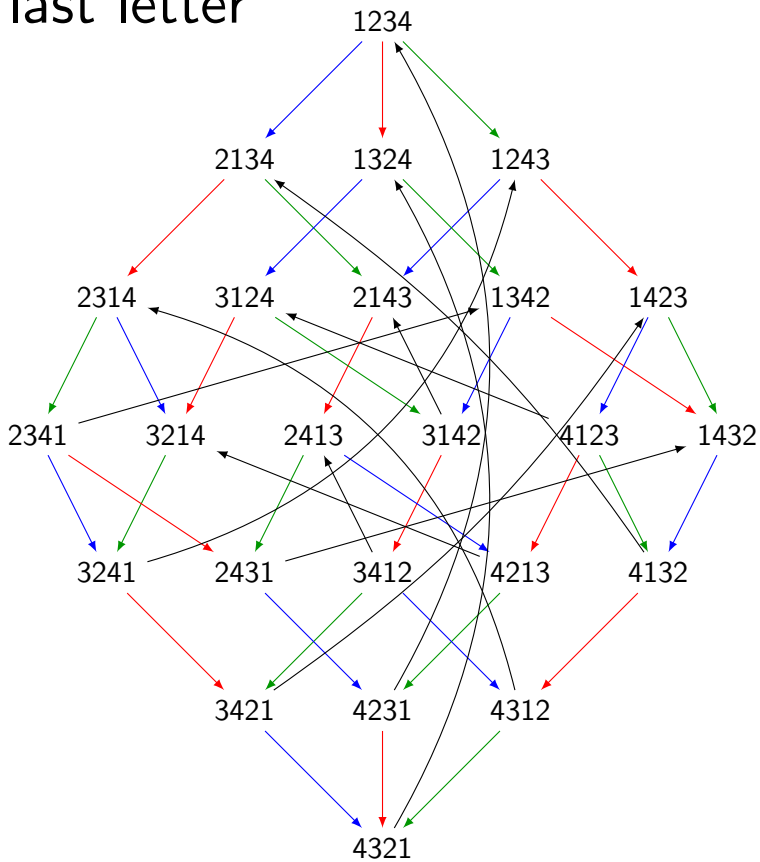

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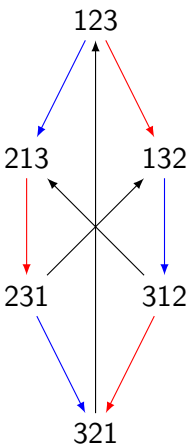

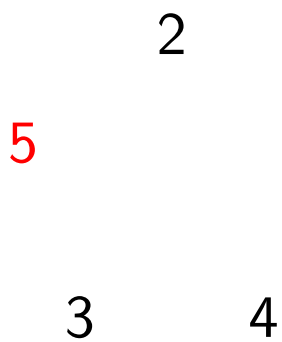

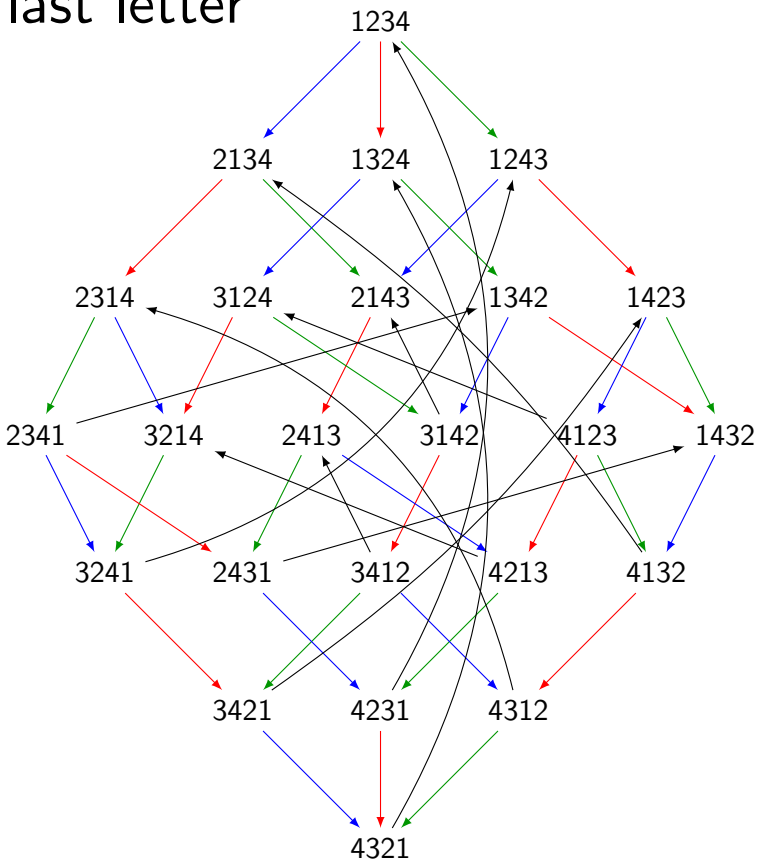

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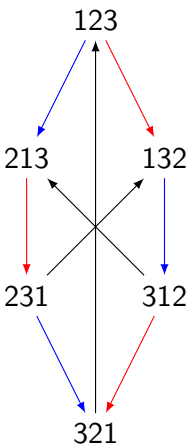

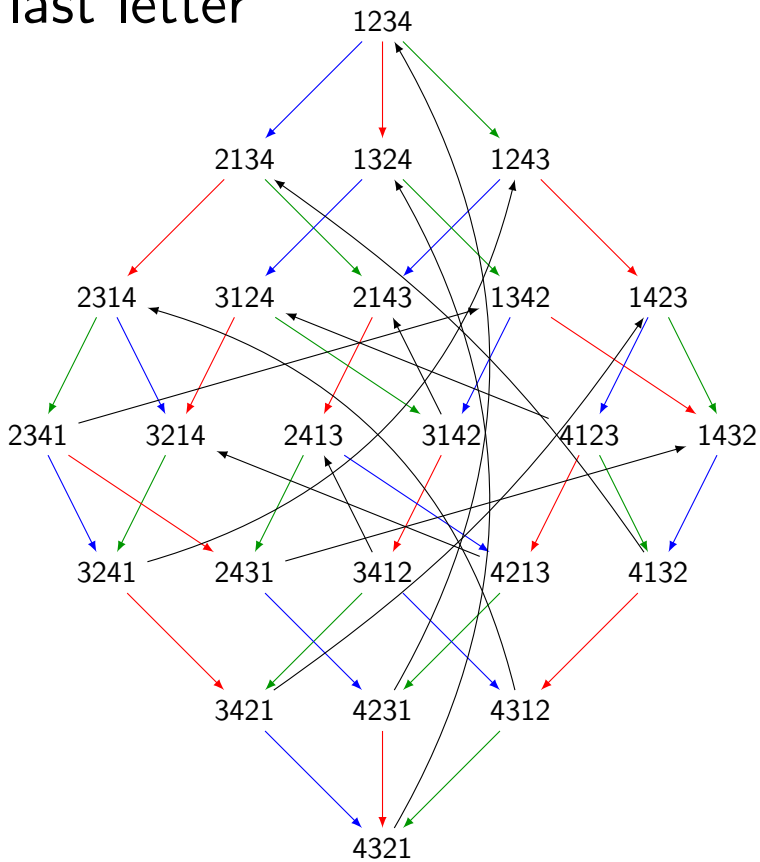

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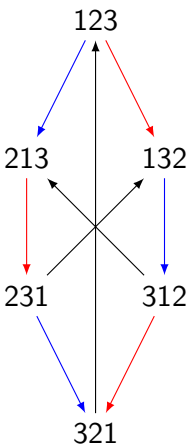

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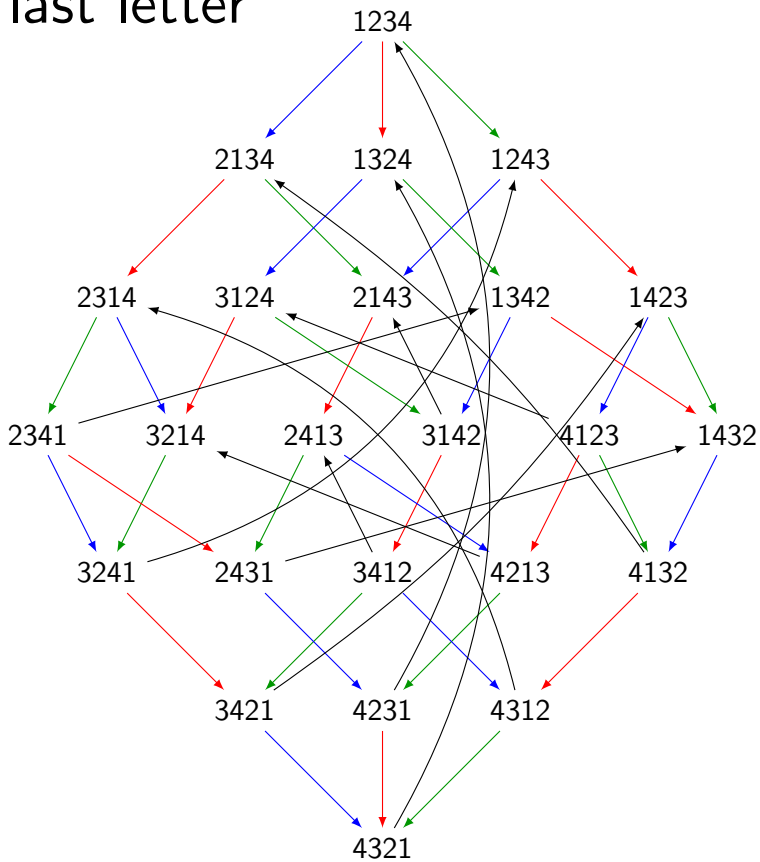

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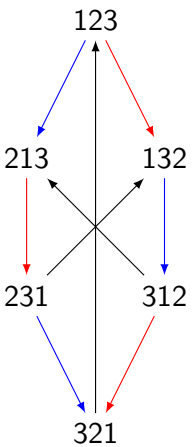

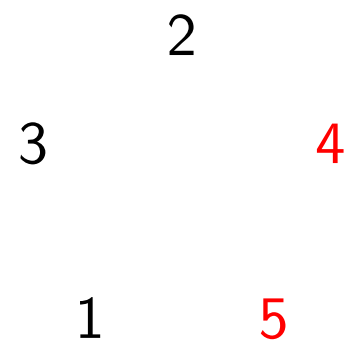

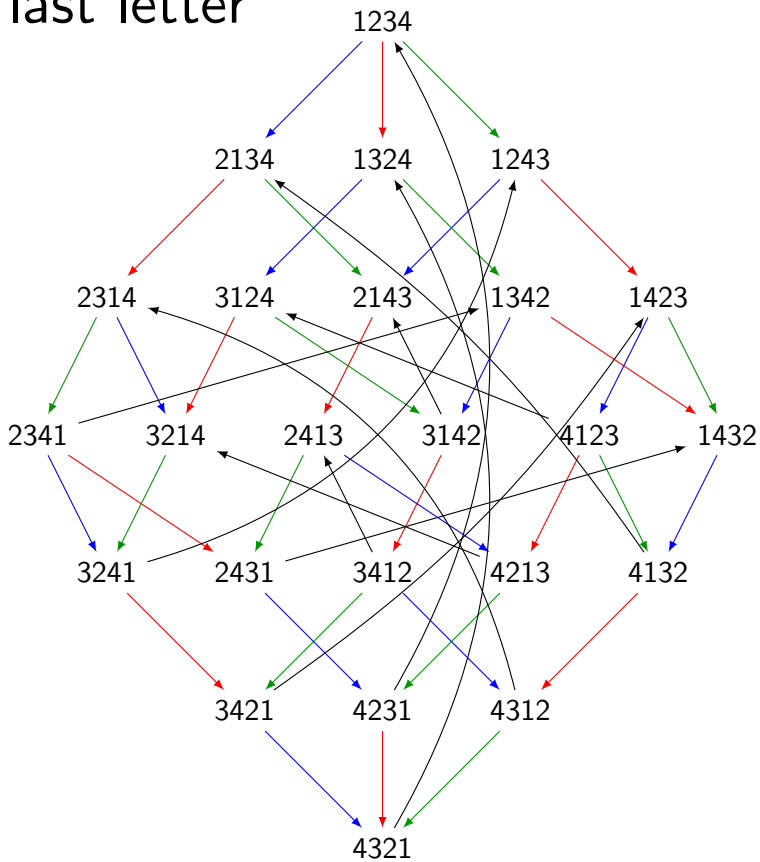

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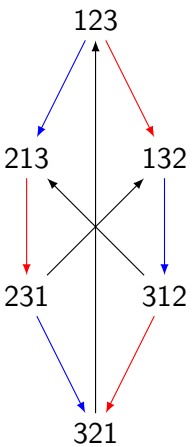

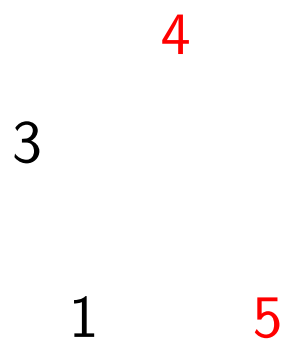

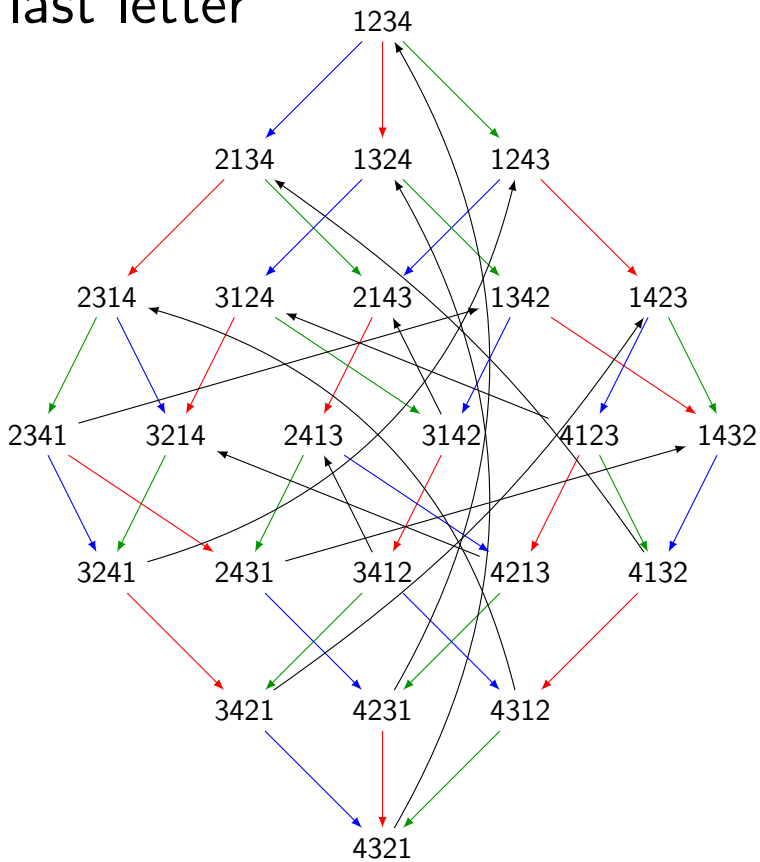

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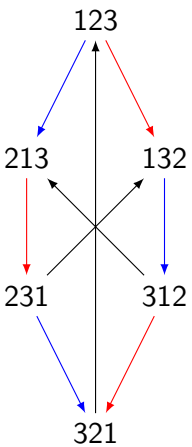

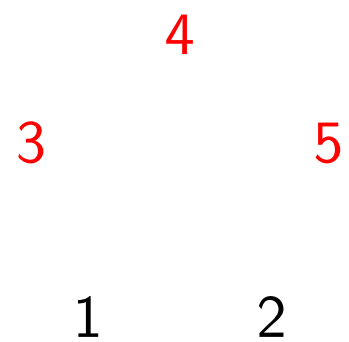

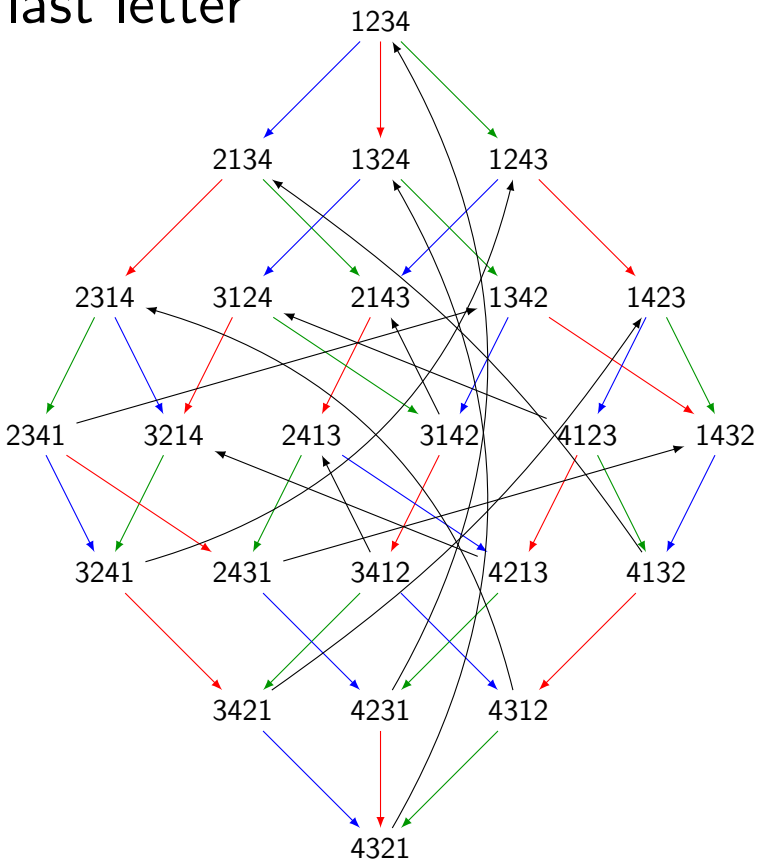

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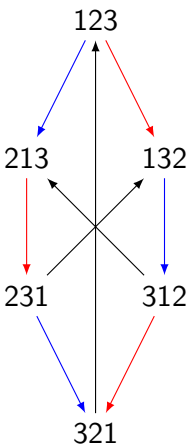

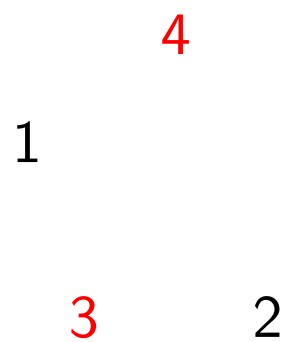

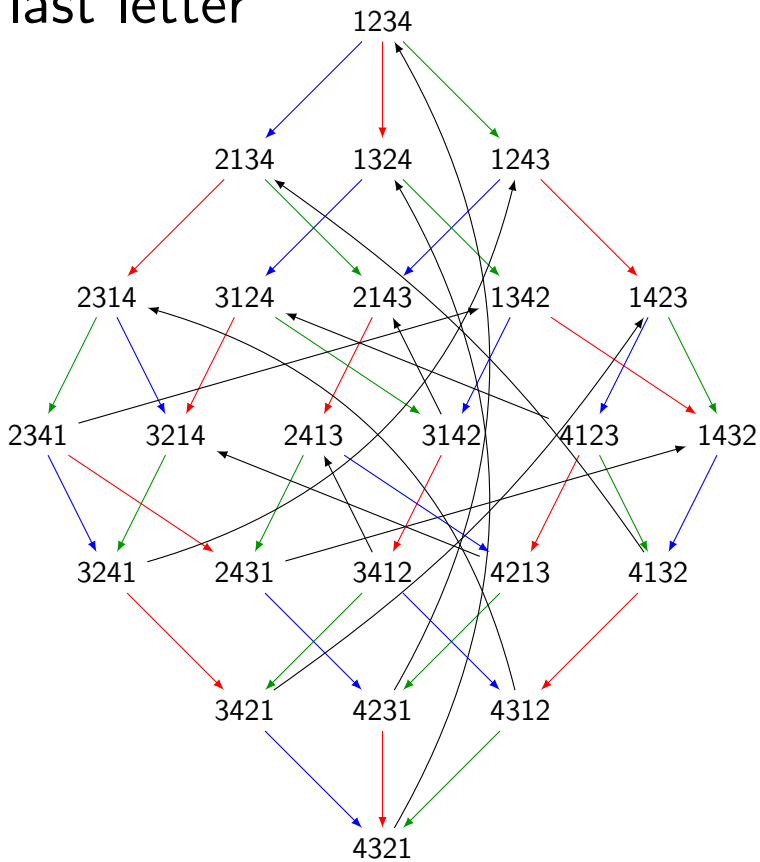

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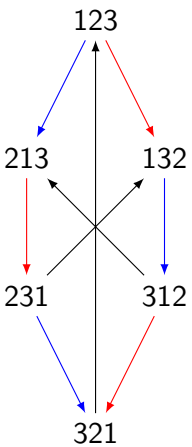

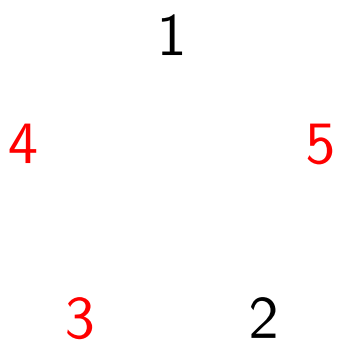

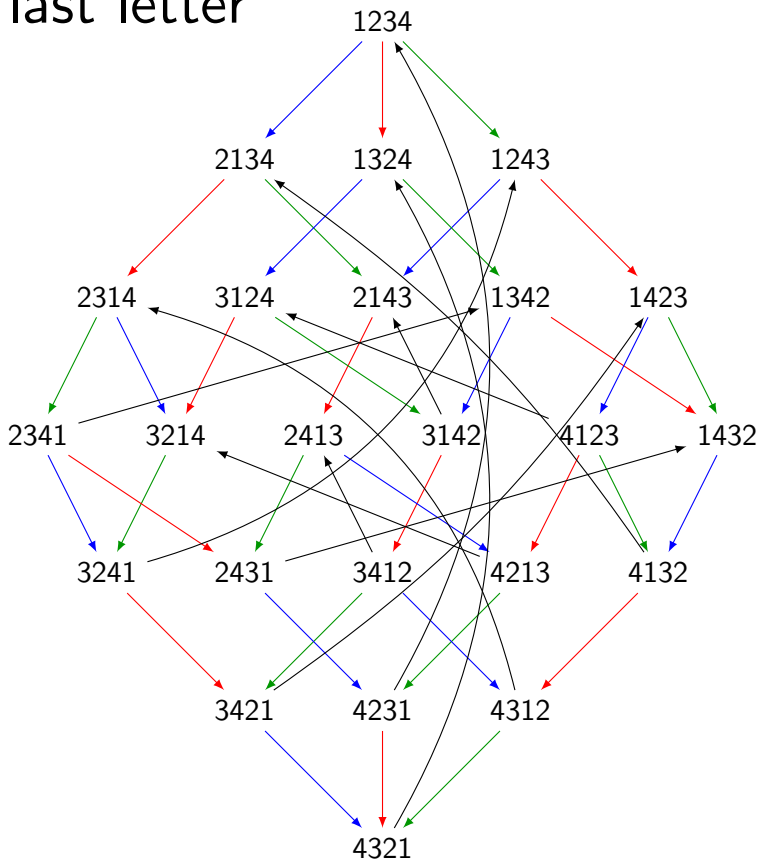

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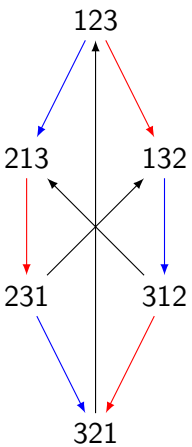

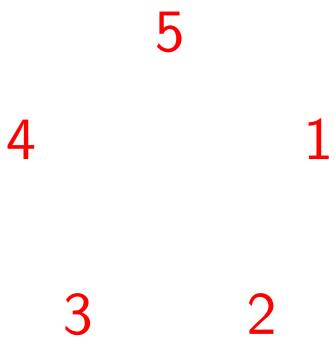

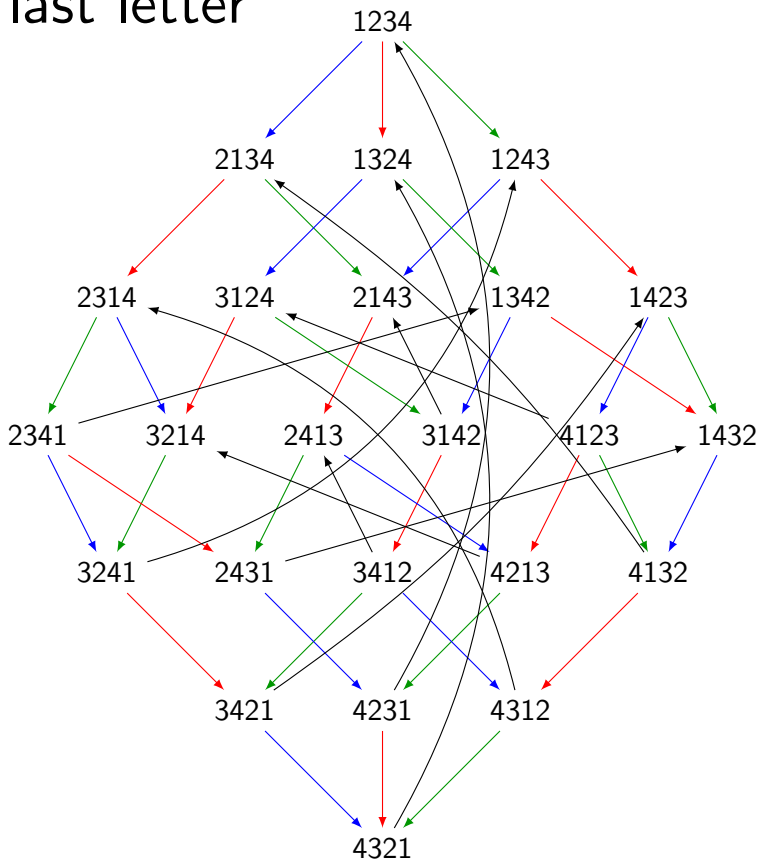

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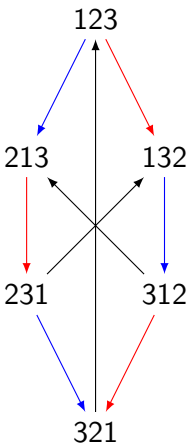

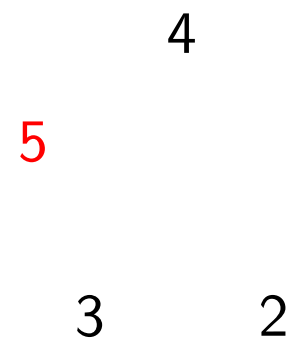

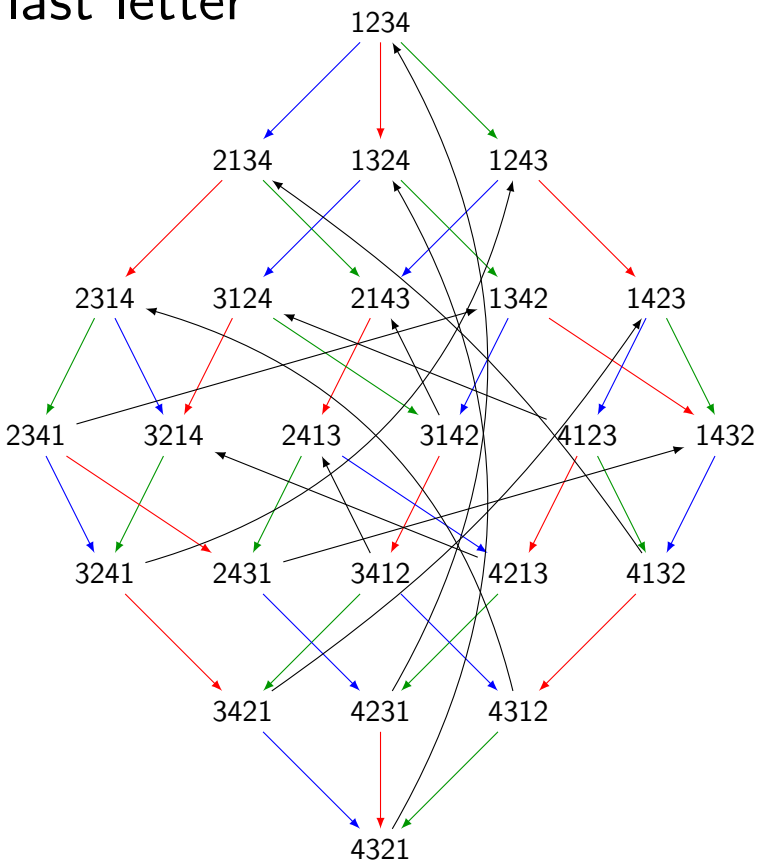

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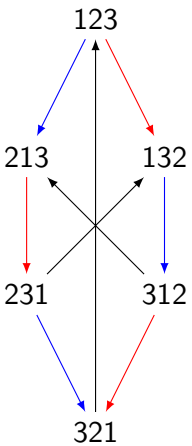

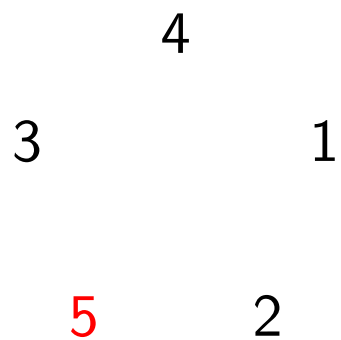

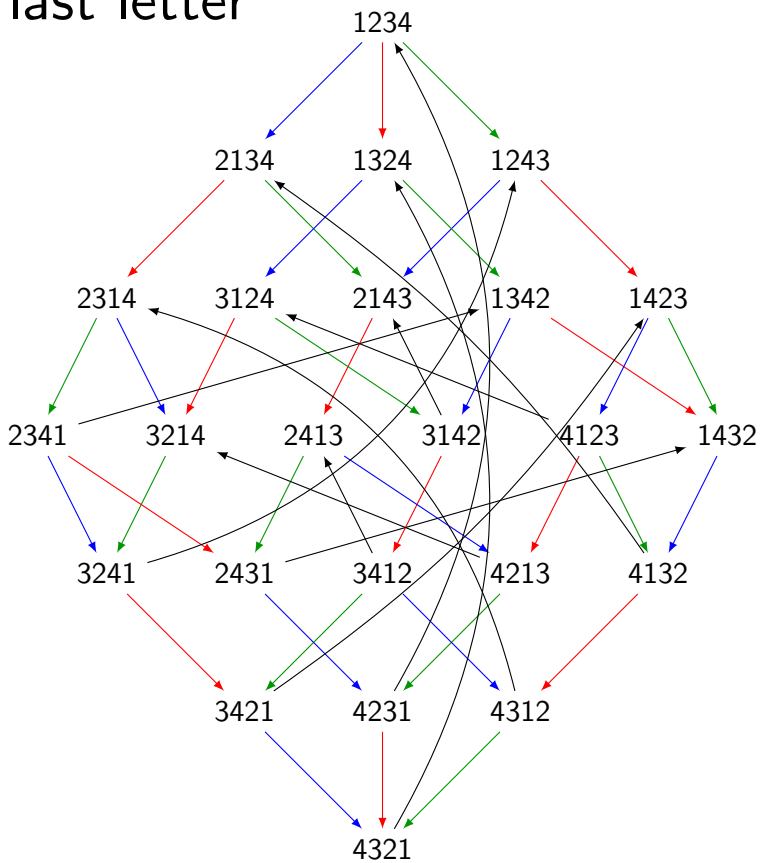

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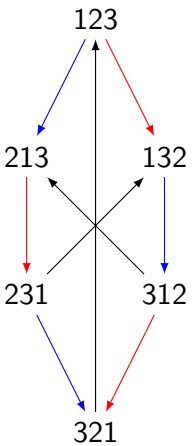

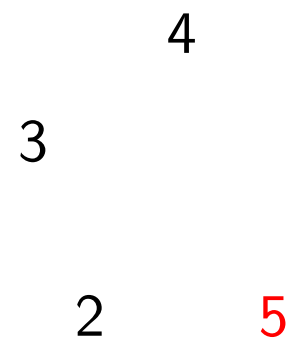

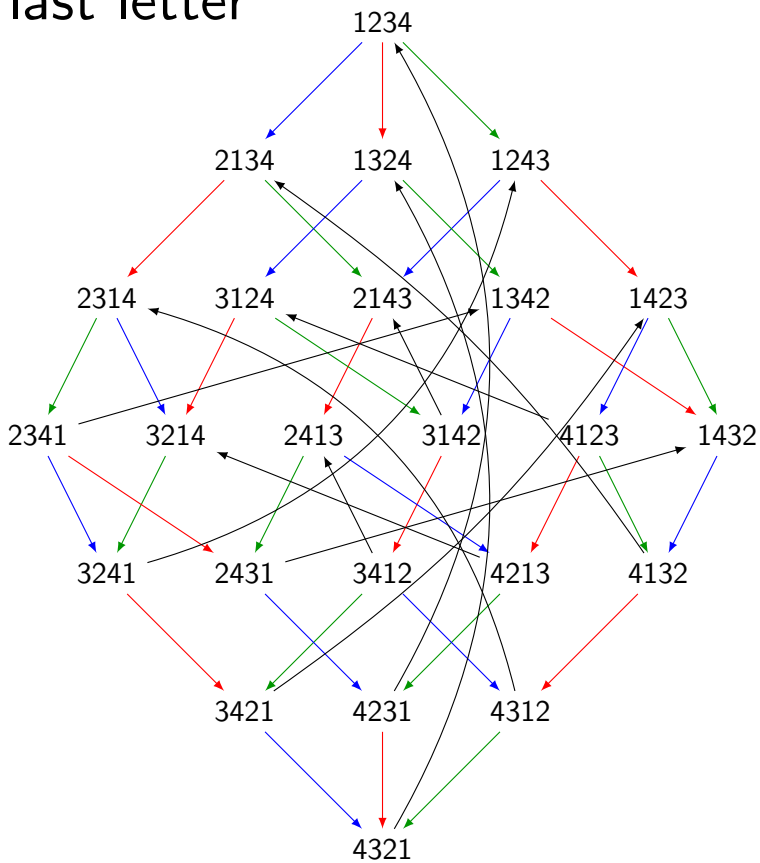

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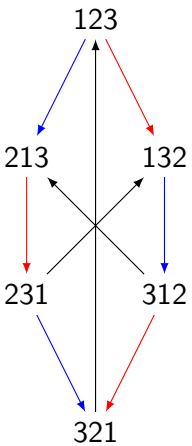

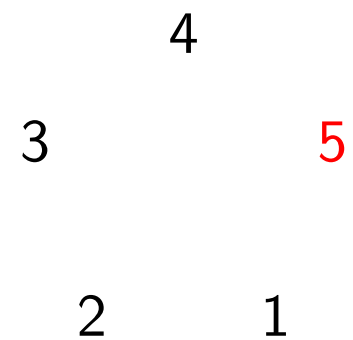

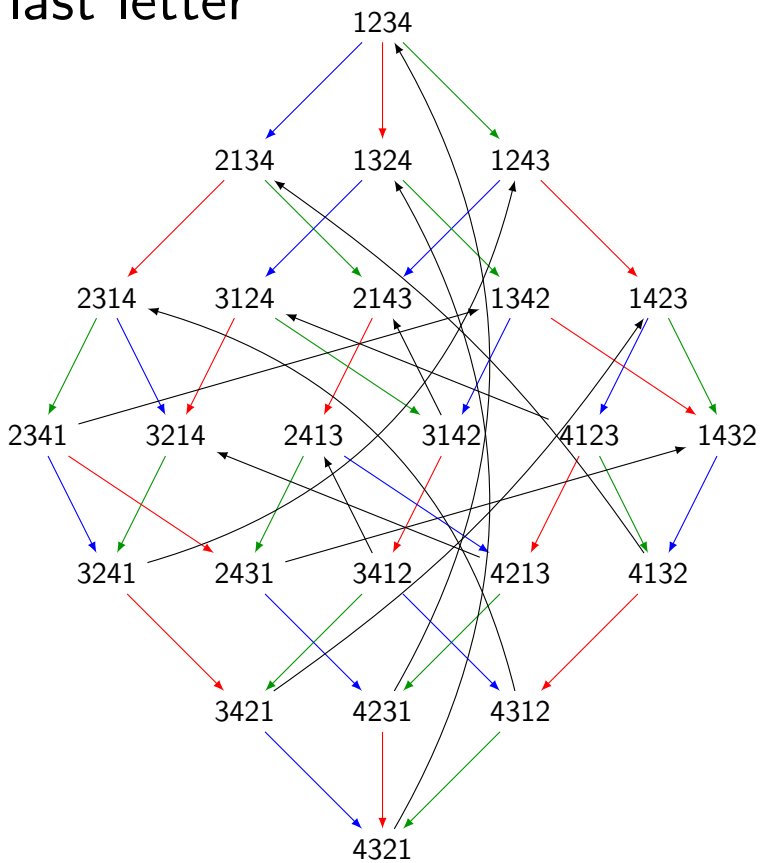

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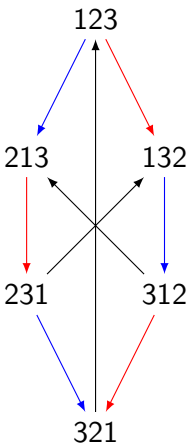

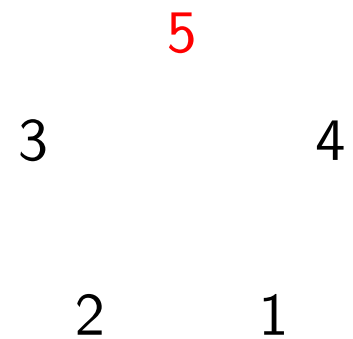

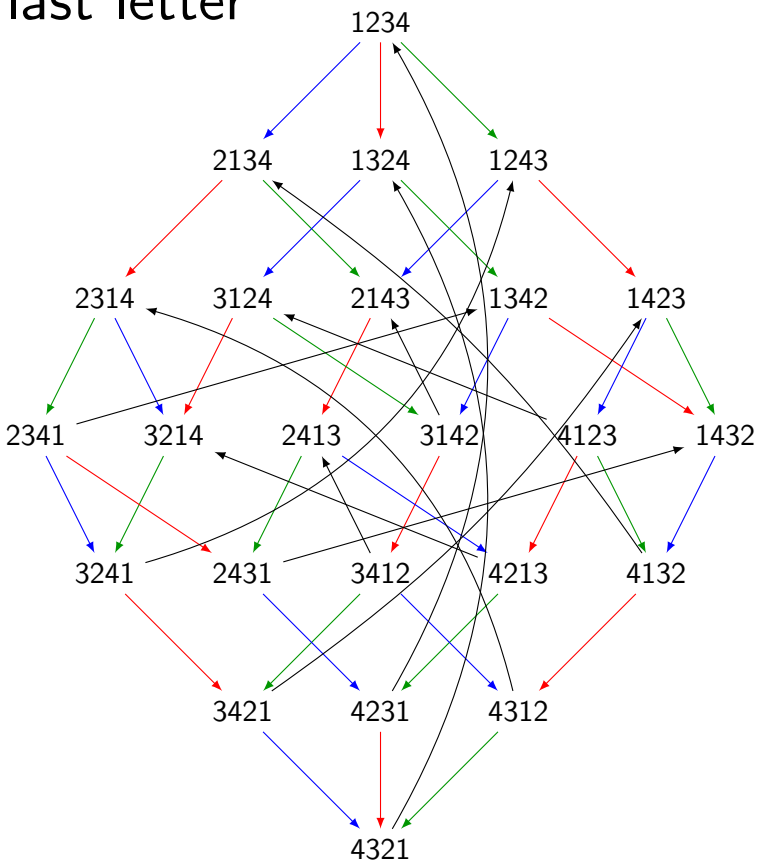

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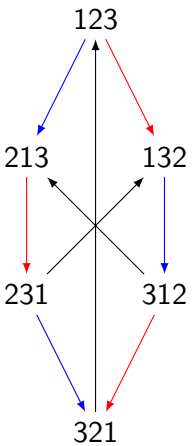

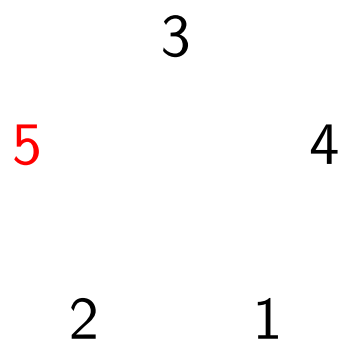

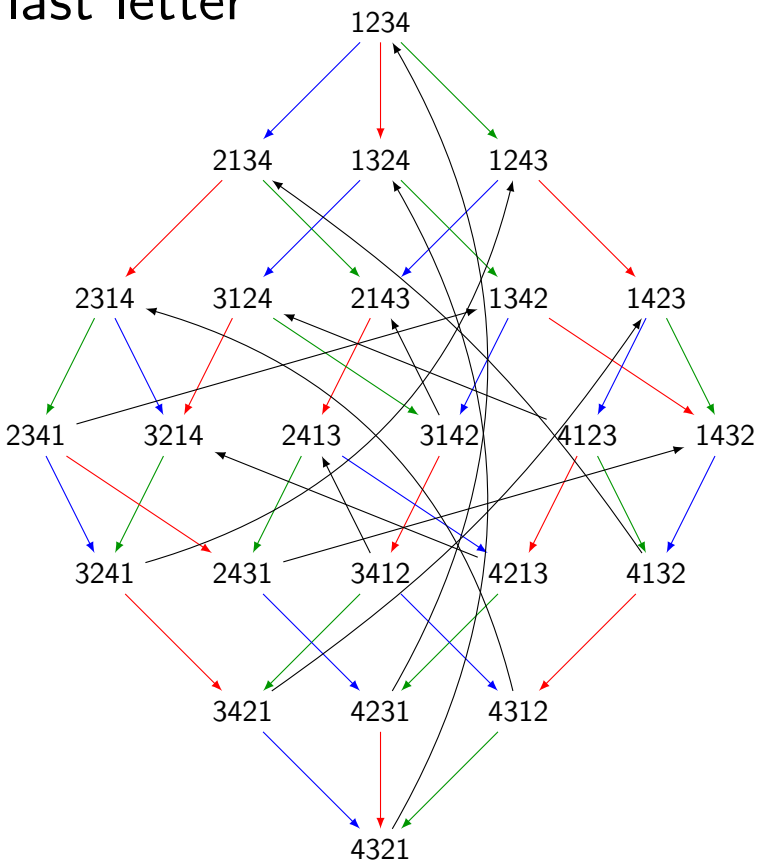

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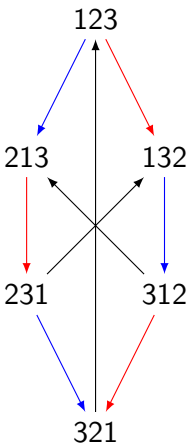

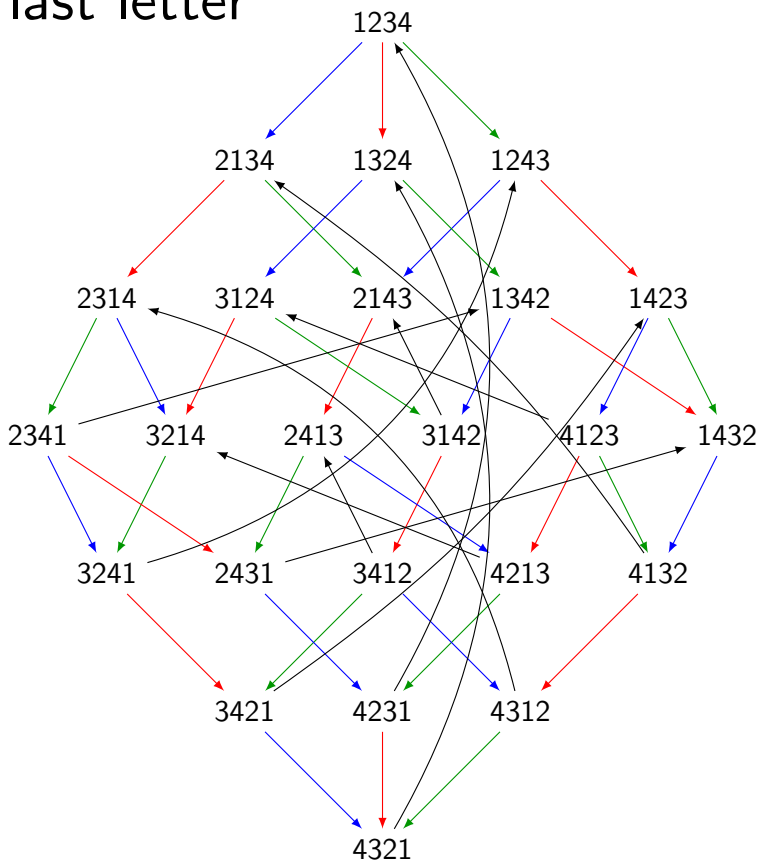

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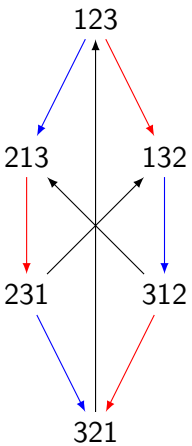

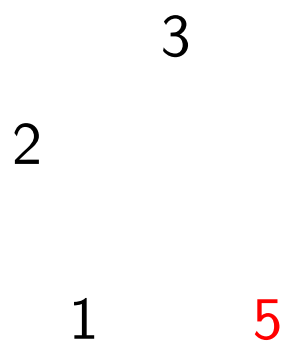

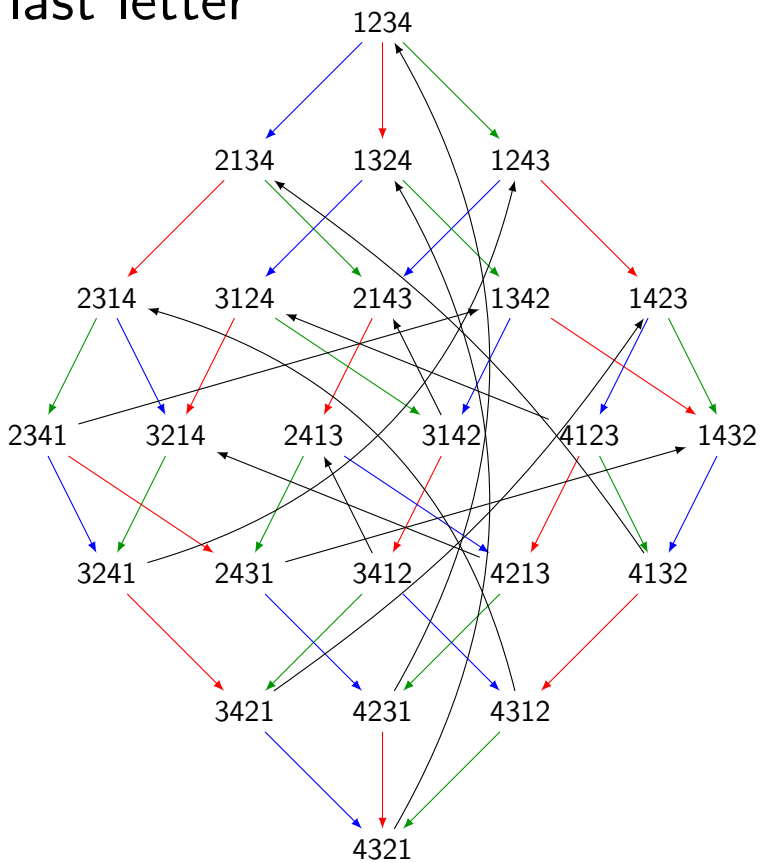

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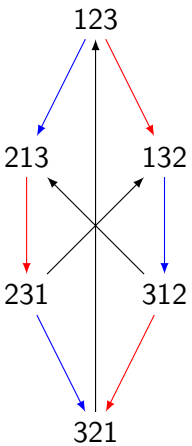

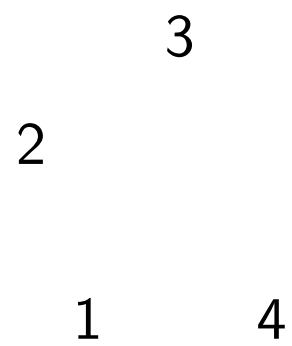

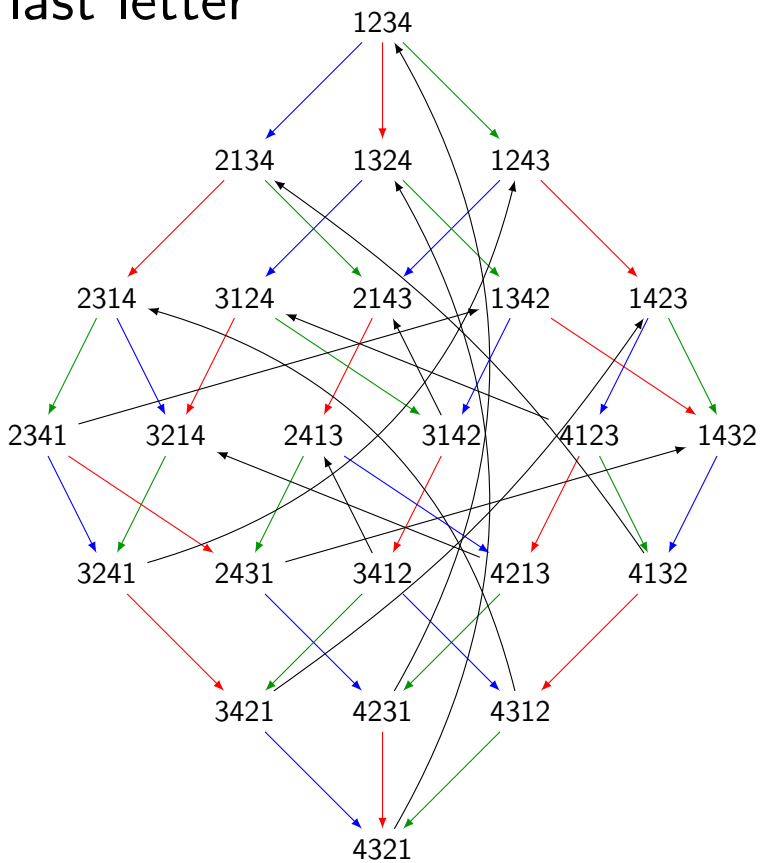

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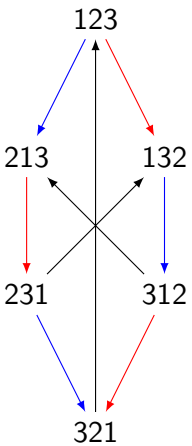

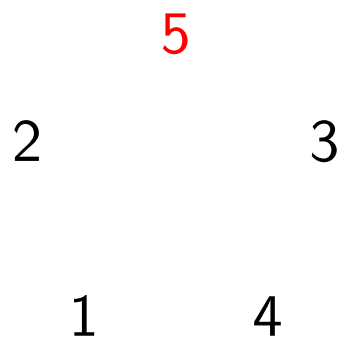

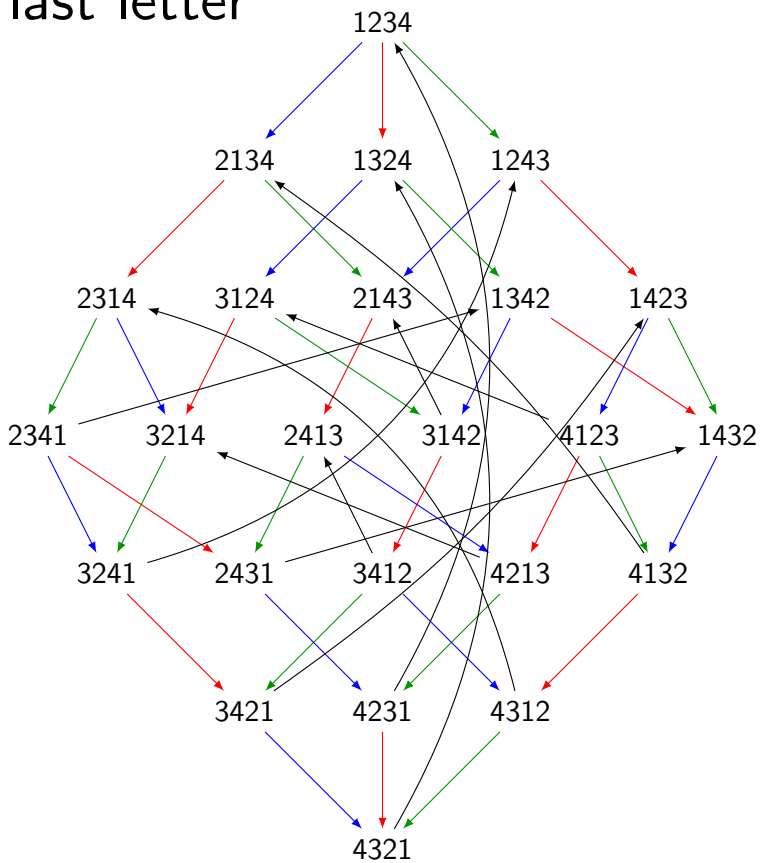

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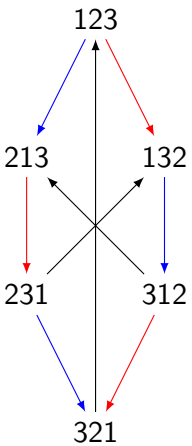

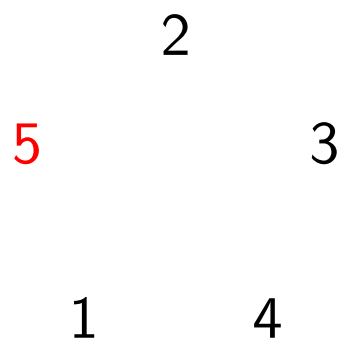

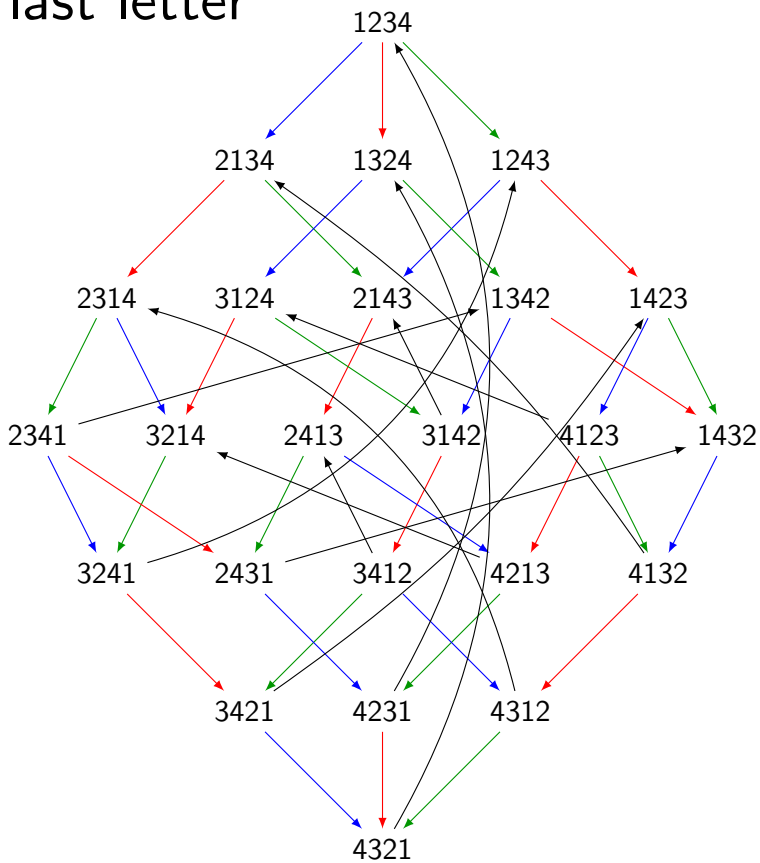

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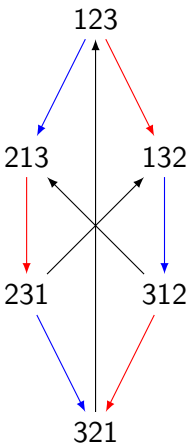

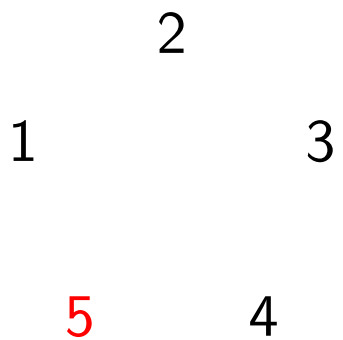

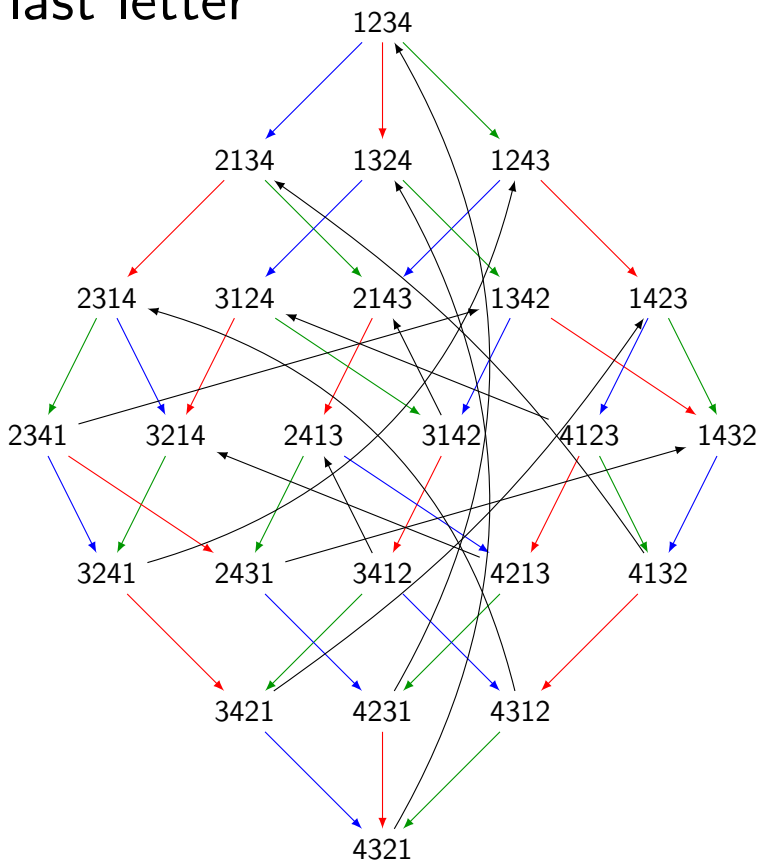

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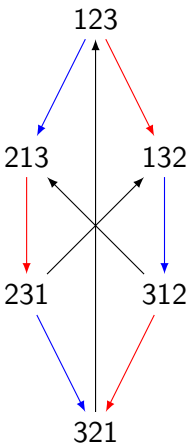

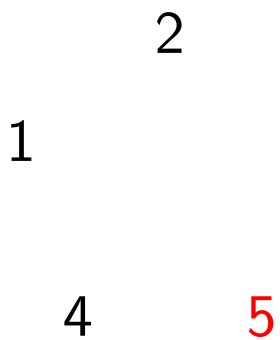

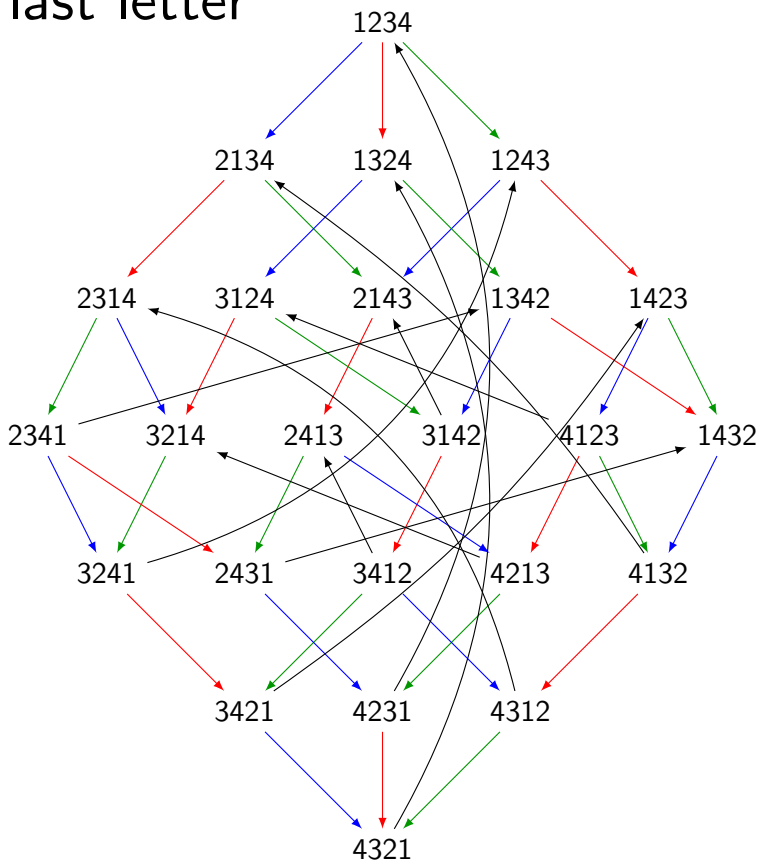

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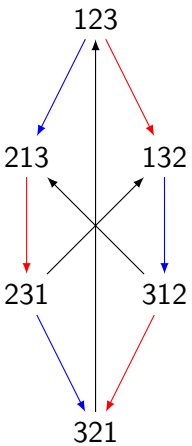

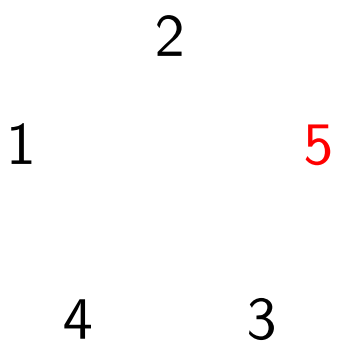

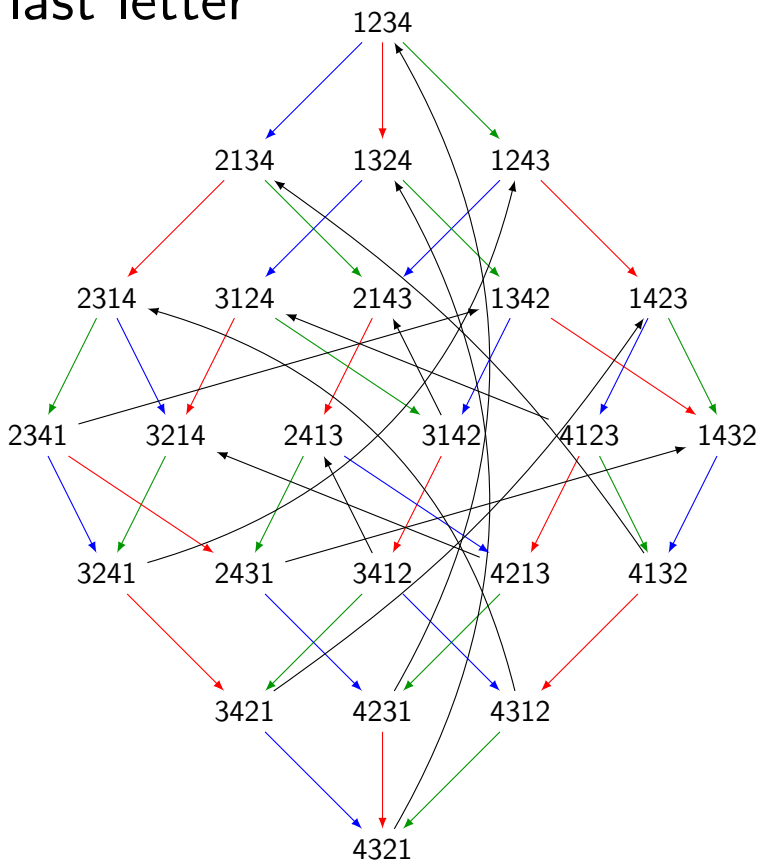

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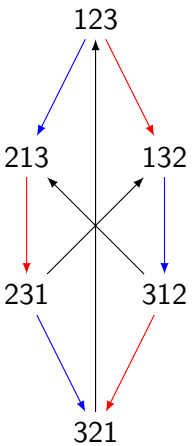

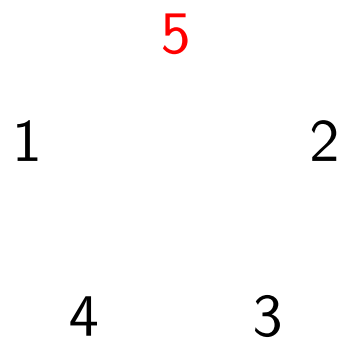

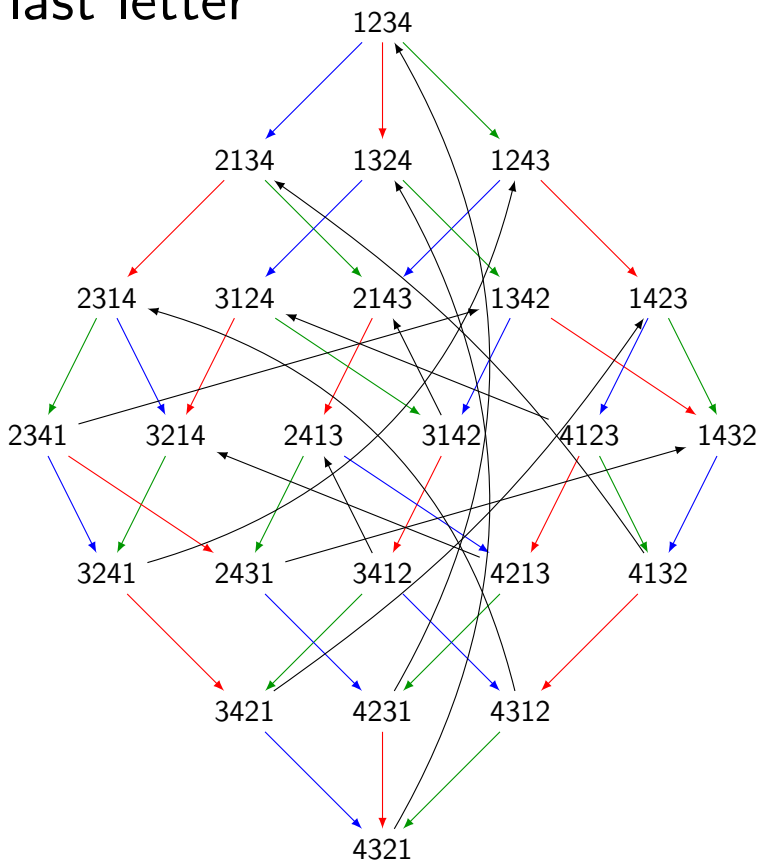

Proposition (HT'08)

$\pi_0, \pi_1, \pi_2, \pi_3, \ldots$ act transitively on permutations

Cyclic bubble sort

$\pi_1, \pi_2, \pi_3, \ldots$ antisort 12345 into 54321 There is no return!

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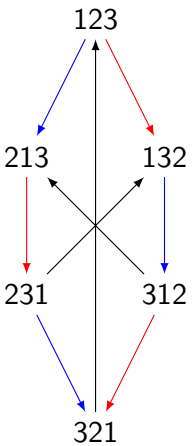

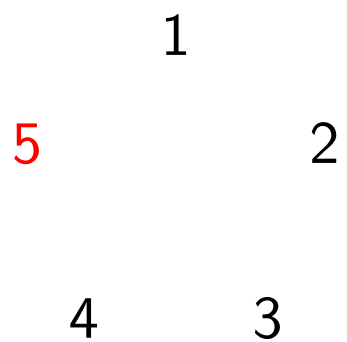

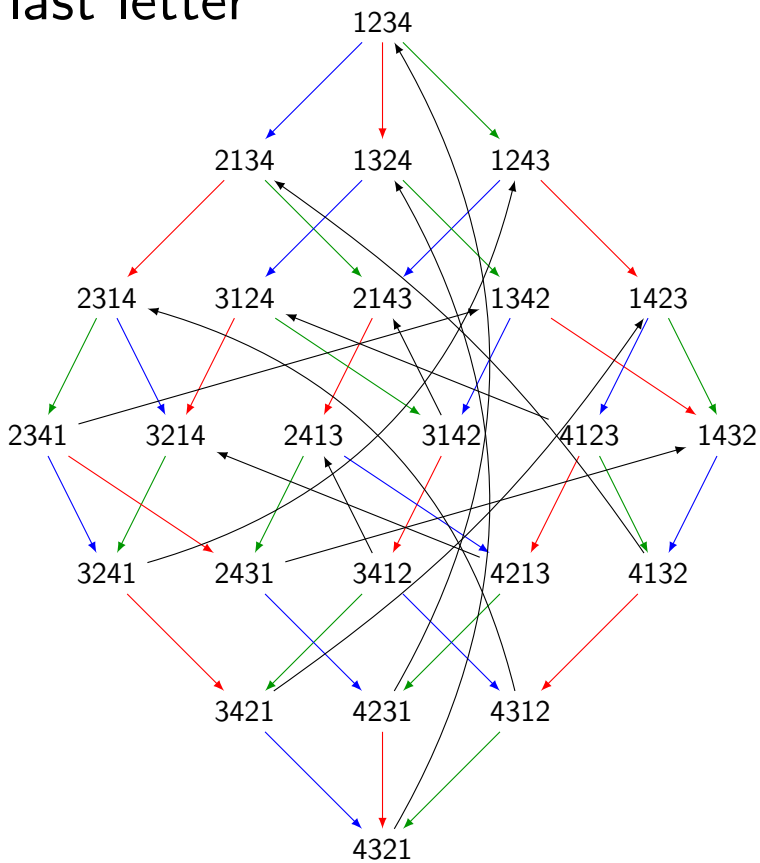

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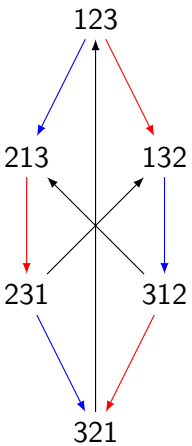

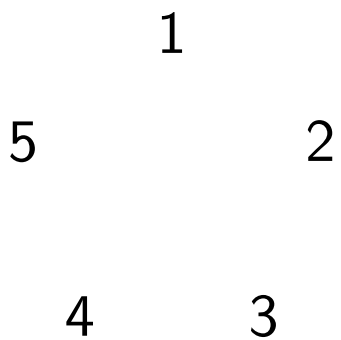

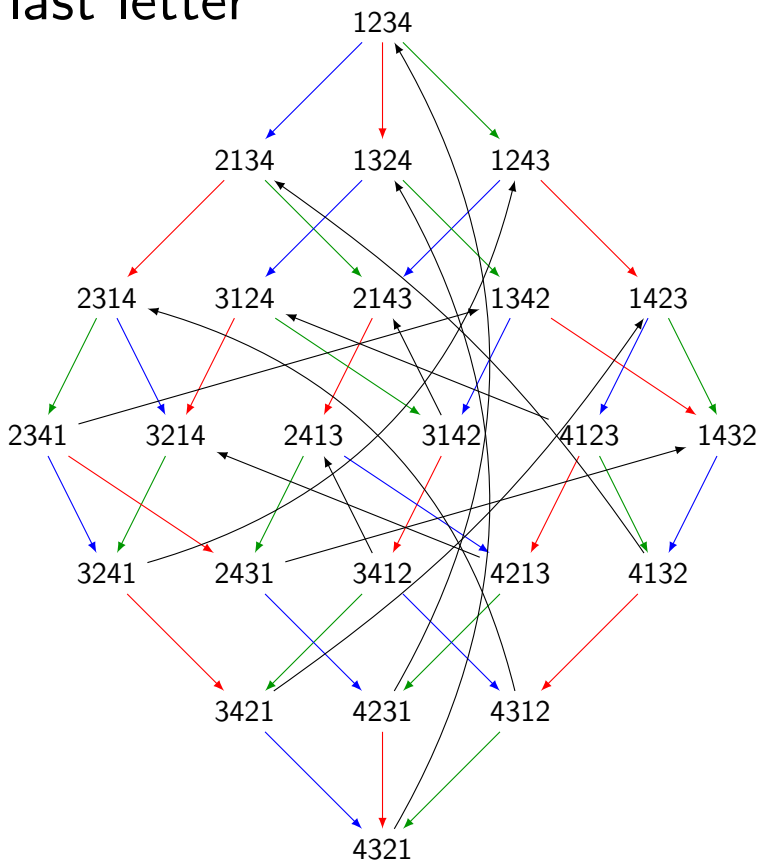

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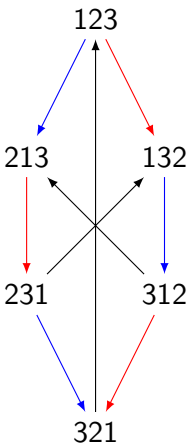

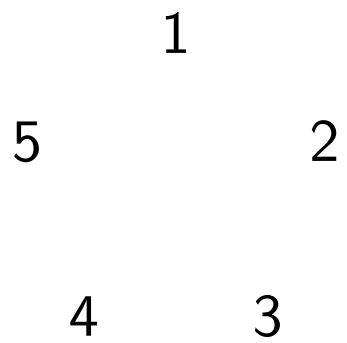

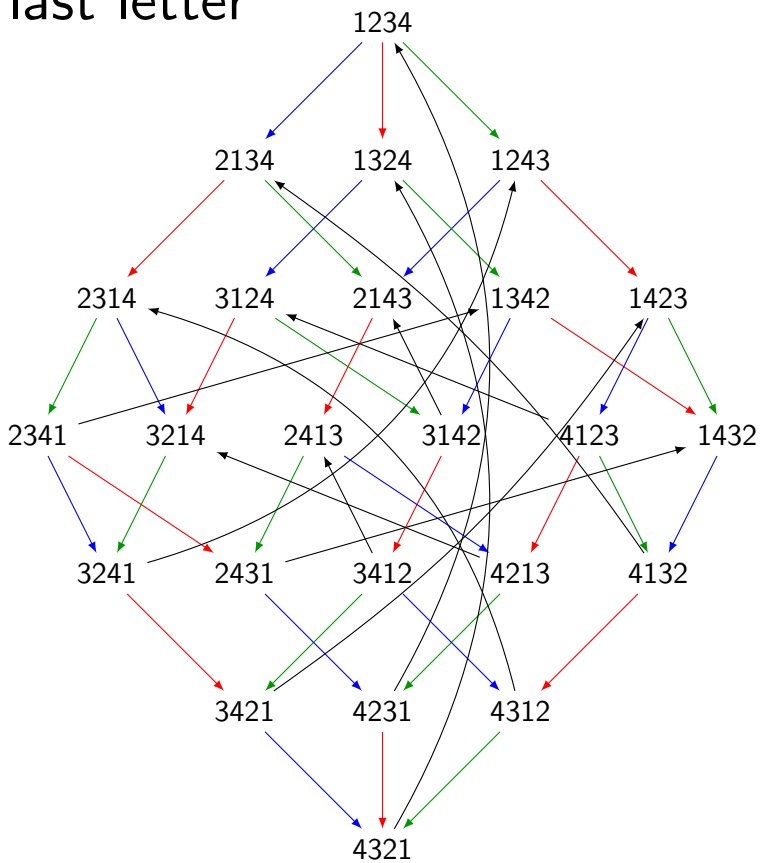

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$\pi_0, \pi_1, \pi_2, \pi_3, \ldots$ *act transitively on permutations*

Coxeter groups

Definition (Coxeter group W)

Generators : $(s_i)_{i \in S}$ (simple reflections)
Relations: $s_i^2 = 1$ and $\underbrace{s_i s_j \cdots}_{m_{i,j}} = \underbrace{s_j s_i \cdots}_{m_{i,j}}$, for $i \neq j$

Group algebra: $\mathbb{C}[W]$

Example (Type A_n: symmetric group $\mathfrak{S}_{n+1}$)

Generators: $(s_i)_{i=1,\ldots,n}$ (elementary transpositions)
Relations:

$$\begin{aligned} s_i^2 &= 1 && \text{for all } 1 \leq i \leq n, \\ s_i s_j &= s_j s_i && \text{for all } |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for all } 1 \leq i \leq n-1. \end{aligned}$$

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Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: $\begin{matrix}0\\1\end{matrix} > 2 - 3 \Rightarrow 4$ $\qquad 1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$
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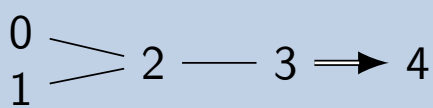
- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

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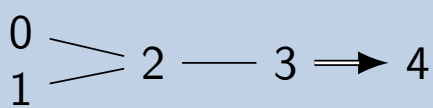
- Type B: 

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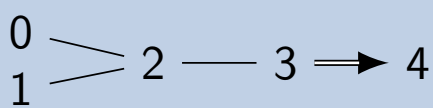
- Type B: 

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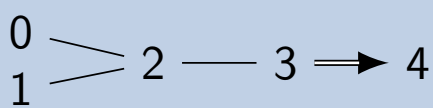
- Type B: 

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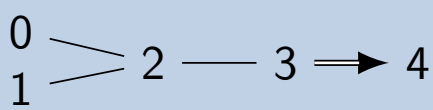
- Type B: 

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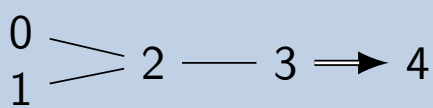
- Type B: 

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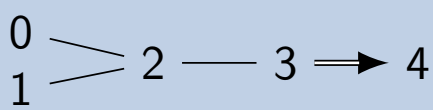
- Type B: 

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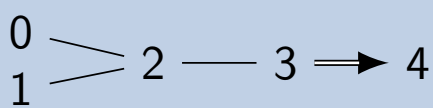
- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{34}21$$

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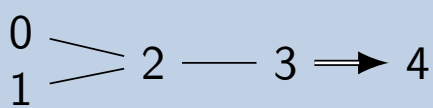
- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

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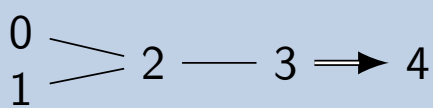
- Type B:  $1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$4321$$

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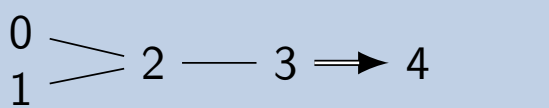
- Type B: 

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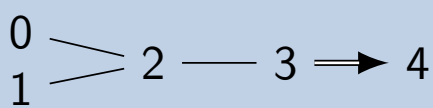
- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$43\underline{1}2$$

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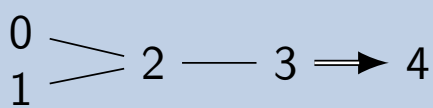
- Type B: 

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$$41\underline{1}32$$

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$$\underline{1}432$$

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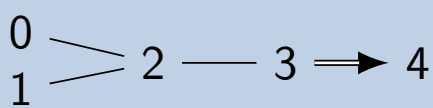
- Type B: 

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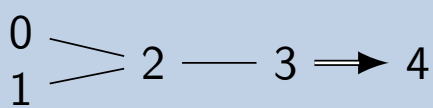
- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{1}43\underline{2}$$

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$$\underline{1}4\underline{2}3$$

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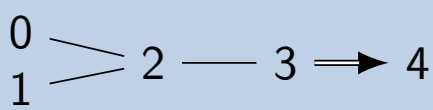
- Type B: 

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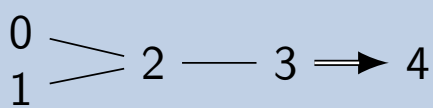
- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

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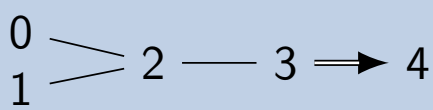
- Type B: 

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$$\underline{12}4\underline{3}$$

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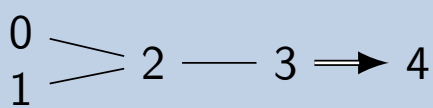
- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$21\underline{3}4$$

- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$21\underline{3}4$$

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Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: $\begin{matrix}0\\1\end{matrix} > 2 - 3 \Rightarrow 4$ $\qquad 1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$2\underline{3}14$$

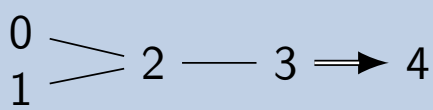
- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$2\underline{3}14$$

- Type-free induction strategy
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Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
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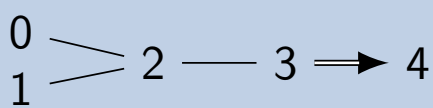
- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{3}214$$

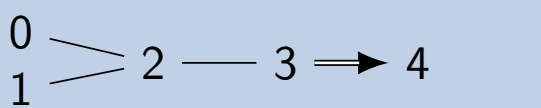
- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{3}214$$

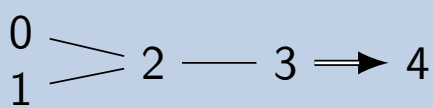
- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{3}241$$

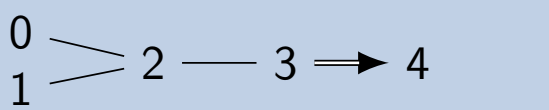
- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{3}24\underline{1}$$

- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: $\overset{0}{1} > 2 - 3 \Rightarrow 4$ $\qquad 1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{3}2\underline{1}4$$

- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

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- *Similar algorithms for types B, C, D*
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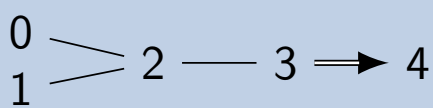
Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{3}\underline{1}24$$

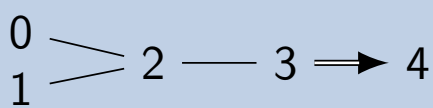
- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{31}24$$

- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: $\begin{matrix}0\\1\end{matrix} > 2 - 3 \Rightarrow 4$ $\qquad 1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$1324$$

- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B:

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

13**2**4

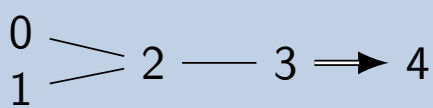
- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$1342$$

- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: $\begin{matrix} 0 \\ 1 \end{matrix} \!\!> 2 - 3 \Rightarrow 4$ $\qquad 1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$1342$$

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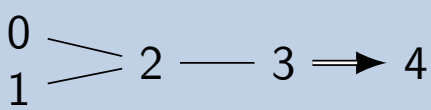
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$13\underline{2}4$$

- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: $\overset{0}{1} > 2 - 3 \Rightarrow 4$ $\qquad 1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$1\underline{2}34$$

- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{2}134$$

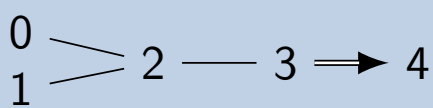
- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{2}134$$

- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
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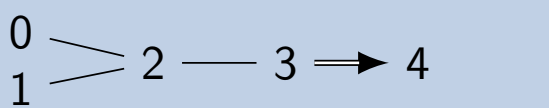
Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{2}314$$

- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
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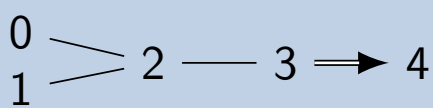
Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{2}341$$

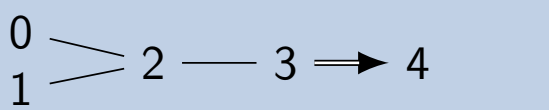
- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{2}34\underline{1}$$

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Recursive sorting algorithms in other types

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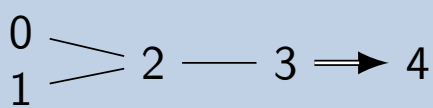
- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{2}3\underline{1}4$$

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- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

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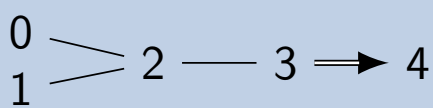
- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{2}\underline{1}34$$

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Proposition (S.,T.,2007)

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Proof

- Type B: $\begin{matrix} 0 \\ 1 \end{matrix} \!\!> 2 - 3 \Longrightarrow 4$ $\qquad 1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$\underline{21}34$$

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Proof

- Type B: $\begin{matrix}0\\1\end{matrix} > 2 - 3 \Rightarrow 4$ $\qquad 1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$1234$$

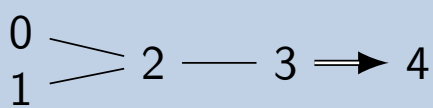
- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B: 

$1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

1234

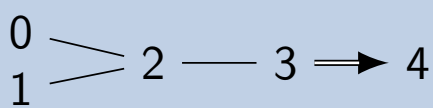
- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
- *Existence for all types (including twisted)*

Proof

- Type B:  $1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$1234$$

- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Recursive sorting algorithms in other types

Proposition (S.,T.,2007)

- *Similar algorithms for types B, C, D*
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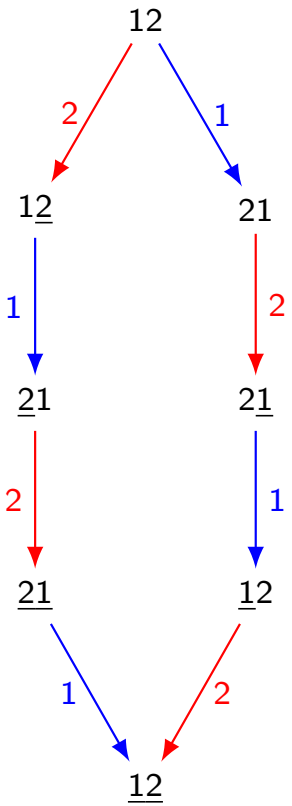
Proof

- Type B: $\overset{0}{1} > 2 - 3 \Rightarrow 4$ $\qquad 1 < 2 < 3 < 4 < \underline{4} < \underline{3} < \underline{2} < \underline{1}$

$$1234$$

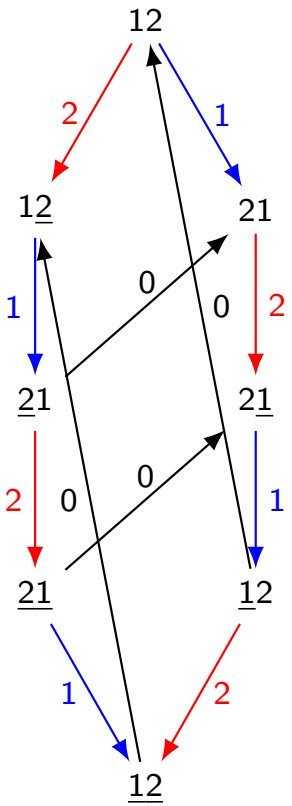
- Type-free induction strategy
- Case by case induction step
  Brute force on computer for the exceptional types (E_8!)

Geometric argument (I)



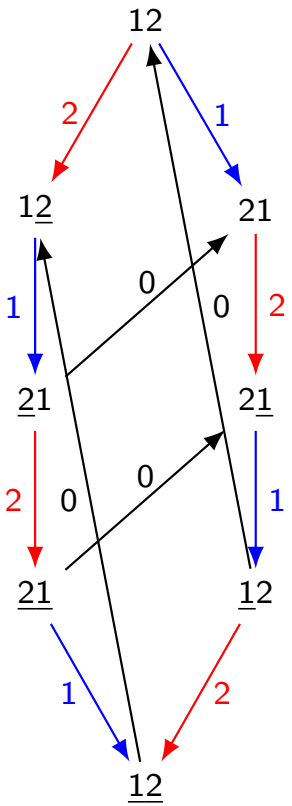
π_0, π_1, π_2 on C_2

Geometric argument (I)

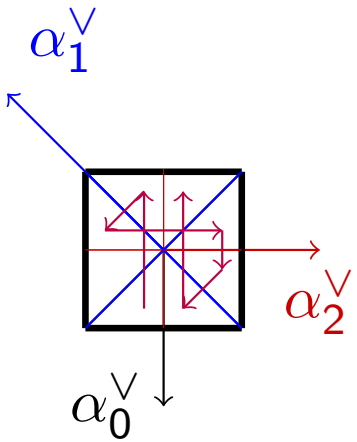


π_0, π_1, π_2 on C_2

Geometric argument (I)

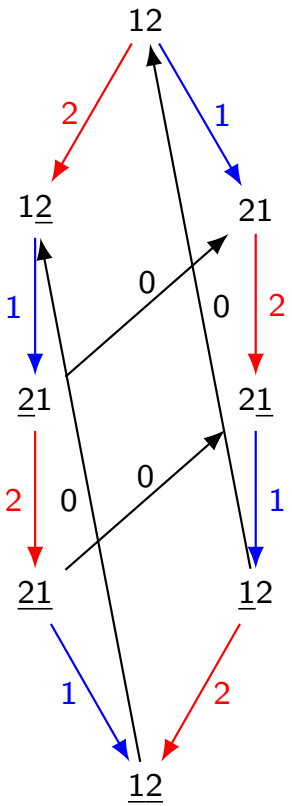


π_0, π_1, π_2 on C_2

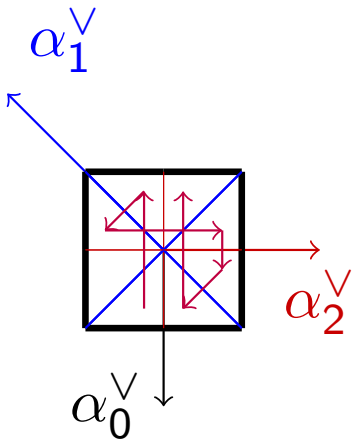


Quotient at level 0
(Steinberg torus)

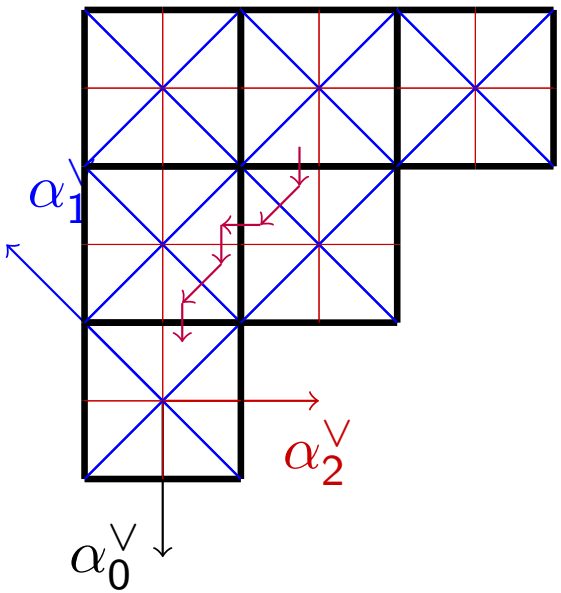
Geometric argument (I)



π_0, π_1, π_2 on C_2

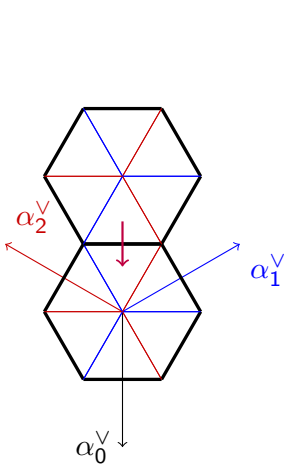


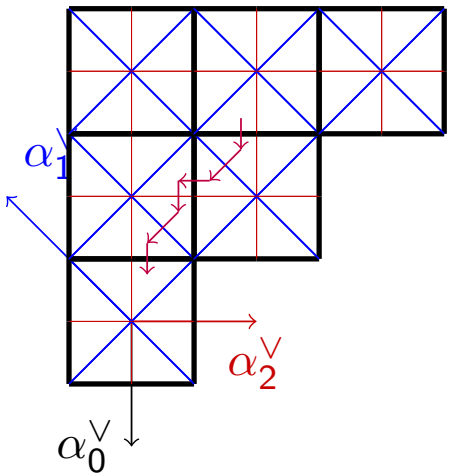
Quotient at level 0
(Steinberg torus)

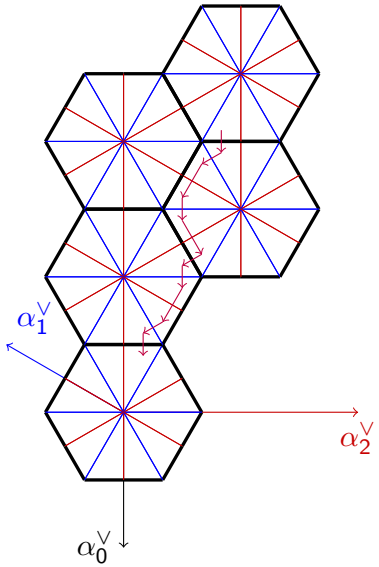


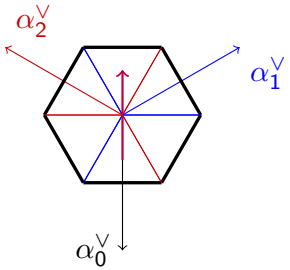
Alcove picture at level 1

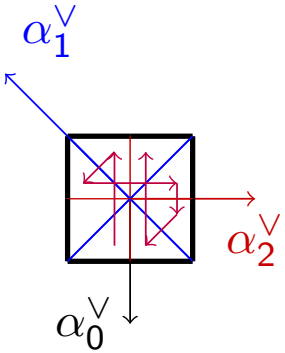
Type free geometric argument (II)

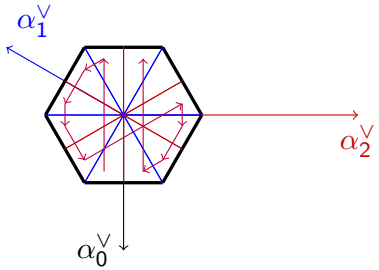












$$1 \overset{0}{\text{---}} 2$$

$$\widetilde{A}_2 = A_2^{(1)}$$

$$0 \xrightarrow{2} 1 \xleftarrow{2} 2$$

$$\widetilde{C}_2 = C_2^{(1)}$$

$$0 \text{ --- } 1 \xrightarrow{3} 2$$

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0-(Iwahori)-Hecke algebras (or monoids)

Definition (0-Hecke algebra H(W)(0))

Generators : $(\pi_i)_{i \in S}$

Relations: $\pi_i^2 = \pi_i$ and $\underbrace{\pi_i \pi_j \cdots}_{m_{i,j}} = \underbrace{\pi_j \pi_i \cdots}_{m_{i,j}}$ for $i \neq j$

Basis: $(\pi_w)_{w \in W}$

Example (Type A_n)

Generators: $(\pi_i)_{i=1,\dots,n}$

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$$\pi_i^2 = \pi_i \quad \text{for all } 1 \leq i \leq n,$$
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Hecke algebras

Take q_1 and q_2 parameters, and set $q := -\frac{q_1}{q_2}$.

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- At $q_1 = q_2 = 0$: nilCoxeter algebra
- At q not 0 nor a root of unity: isomorphic to $\mathbb{C}[W]$

Motivation: Macdonald polynomials, mathematical physics, Schur-Weyl duality for quantum groups, representations of $GL(\mathbb{F}_q)$, and ...

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Geometric questions:

Fl_n variety of complete flags in $\mathbb{C}^n$

$$\emptyset = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$$

with $\dim V_i = i$

Cohomology ring (Borel)

$$H^*(\mathrm{Fl}_n, \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]/I_n$$

with I_n ideal generated by symmetric functions without constant term

Schubert classes $\leftrightarrow$ Schubert polynomials $\mathfrak{S}_w$, $w \in S_n$

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Algebraic definition of Schubert polynomials

Definition (Schubert polynomials, Lascoux-Schützenberger)

$$\mathfrak{S}_w = \partial_{w^{-1}w_0}\left(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}\right)$$

where ∂_i is the divided difference operator

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Combinatorial formulation of Schubert polynomials

Theorem (Fomin, Kirillov)

Coefficient of $x_1^{a_1} \cdots x_n^{a_n}$ in $\mathfrak{S}_w(x_1, \ldots, x_n)$:

number of factorizations of w in the nilCoxeter algebra into factors of length $a_1, \ldots, a_n$ that are strictly decreasing

Replace nilCoxeter algebra by the 0-Hecke algebra
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Characterization, representation theory
Connection with the affine Hecke algebra
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- Find combinatorial models for algebras and representations
- As simple as possible, but no simpler
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- Concrete and effective
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- Sage-Combinat: `combinat.sagemath.org`

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