Théorie des représentations effective des monoïdes

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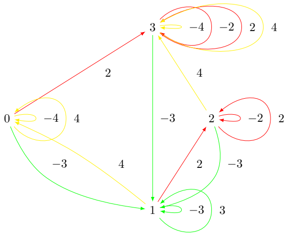

Abstract

Théorie des représentations effective pour les monoides finis
La théorie des représentations des algèbres de dimension finie est un sujet classique. De notre point de vue, c'est un outil puissant pour extraire des données combinatoires depuis une telle algèbre A, et ce d'autant plus qu'il est effectif. En combinatoire algébrique, A est souvent l'algèbre d'un semigroupe ou d'un monoïde M. Cette information peut-elle permettre de mieux comprendre ou au moins calculer la théorie des représentations, comme c'est le cas pour les groupes?
C'est un sujet en pleine effervescence et nous en présenterons quelques aspects. En particulier, nous montrerons comment exprimer la matrice des invariants de Cartan de l'algèbre de M sur un corps K de caractéristique zéro à l'aide de caractères et d'une statistique combinatoire simple. En particulier, elle peut être calculée efficacement. Lorsque M est apériodique, cette approche se généralise à toute caractéristique et aux anneaux principaux comme Z.

L'exposé s'appuiera sur des exemples concrets, la plupart liés aux groupes de Coxeter;

la démarche exploratoire sera illustrée par quelques calculs typiques avec le logiciel

Sage.

Combinatorial Representation Theory I

Representation theory: a tool to extract combinatorics from algebras

- dimension of simple and indecomposable projective modules
($\mathfrak{S}[n], \mathrm{gl}_n$: Kostka numbers)
- induction and restrictions multiplicities
($\mathfrak{S}[m] \times \mathfrak{S}[n] \to \mathfrak{S}[m+n]$: Littlewood-Richardson rules)
- Cartan invariant matrices and quivers
($H_n(0)$: counting permutation by descents and recoils)
- decomposition map
($H_n(q \mapsto 0)$: counting tableaux by shape and descents)

Combinatorial Representation Theory II

Mostly effective: computer exploration !

Depending on

- the base field ($\mathbb{Q}$ or some extension)
- the sparsity of the multiplication table
- ...

Dimension up to 50 to 2000

Several recent examples are monoid algebras

- 0-Hecke algebras (Norton, Carter, Krob-Thibon, Duchamp-Hivert-Thibon, Fayers, Denton)
- Non-decreasing parking function (Denton-Hivert-Schilling-T)
- Solomon-Tits algebras (Schocker, Saliola)
- Left Regular Bands (Brown) ...

...but this fact is seldom used ...

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Running example I: Order preserving functions on the chain

Definition

$f : \{1, \ldots, n\} \longmapsto \{1, \ldots, n\}$ is **order preserving** if:

$$i \leq j \quad \Longrightarrow \quad f(i) \leq f(j)$$

Example

The order preserving functions on $\{1 < 2 < 3\}$:

$$\{111, 112, 113, 122, 123, 133, 222, 223, 233, 333\}$$

Remark

If f, g are order preserving, then so is fg.
Hence, the set $\mathcal{O}_n$ of such functions is a **monoid** *!*

This still works if $\leq$ is replaced by a partial order

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The biHecke monoid of a Coxeter group

Motivation: Schubert calculus, symmetric function

Divided differences operators:

$$\partial_i f(x_1, \dots, x_n) := \frac{f(\dots, x_i, x_{i+1}, \dots) - f(\dots, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}}$$

$$\pi_i := \partial_i x_i, \qquad \hat{\pi}_i := 1 - \pi_i, \qquad s_i, \qquad \dots$$

Applications

- (Algorithmic) exploitation of symmetries
- Schur, Schubert, Macdonald, Kazhdan-Lusztig polynomials, (affine) Stanley symmetric functions
- Mathematical physics, Representation theory, Probability, ...
- Schur-Weyl duality for quantum groups, representations of $GL(\mathbb{F}_q)$, ...

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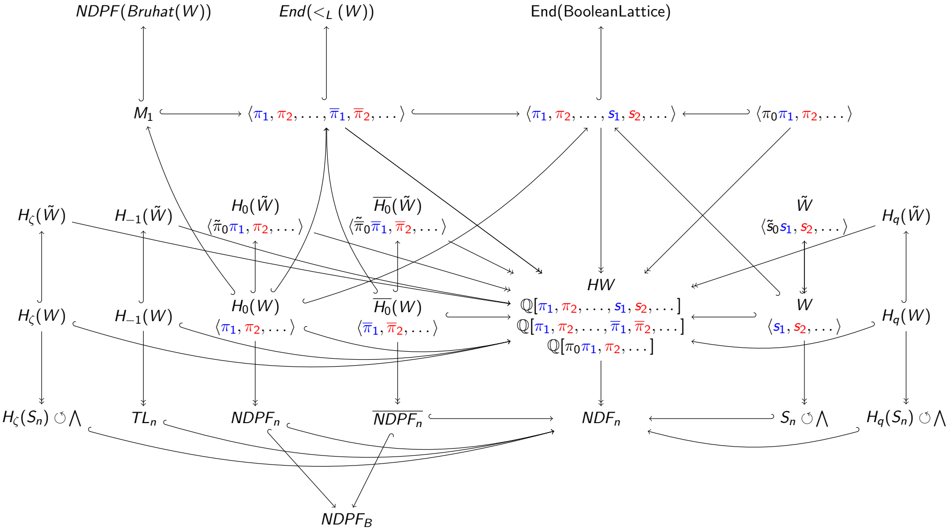
The Big Picture



Coxeter group

Definition (Coxeter group W)

Generators : $s_1, s_2, \ldots$ (simple reflections)

Relations: $s_i^2 = 1$ and $\underbrace{s_i s_j \cdots}_{m_{i,j}} = \underbrace{s_j s_i \cdots}_{m_{i,j}}$, for $i \neq j$

0-Hecke monoid

Definition (0-Hecke monoid $H_0(W)$ of a Coxeter group W)

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Theorem

$|H_0(W)| = |W|$
+ *lots of nice properties*

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The biHecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots \rangle$

$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Theorem (Hivert, Schilling, T '08)

$M(W)$ admits $|W|$ simple / indecomposable projective modules

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w . \dim P_w$$

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Anatomy of a monoid

Understanding the multiplication?

First approach: the multiplication table:

$*$	111	112	113	122	123	133	222	223	233	333
111	111	111	111	111	111	111	222	222	222	333
112	111	111	111	112	112	113	222	222	223	333
113	111	112	113	112	113	113	222	223	223	333
122	111	111	111	122	122	133	222	222	233	333
123	111	112	113	122	123	133	222	223	233	333
133	111	122	133	122	133	133	222	233	233	333
222	111	111	111	222	222	333	222	222	333	333
223	111	112	113	222	223	333	222	223	333	333
233	111	122	133	222	233	333	222	233	333	333
333	111	222	333	222	333	333	222	333	333	333

The Cayley graph of a monoid

Remark

Thanks to associativity, it is sufficient to consider products

$$xg, \quad \textit{for } x \in M \textit{ and } g \textit{ a generator}$$

Definition (Cayley graph)

Graph with edges $x \overset{g}{\longmapsto} xg$

Example

Canonical generators for $\mathcal{O}_3$:

$$\pi_1^+ = 223, \qquad \pi_2^+ = 133$$
$$\pi_1^- = 113, \qquad \pi_2^- = 122$$

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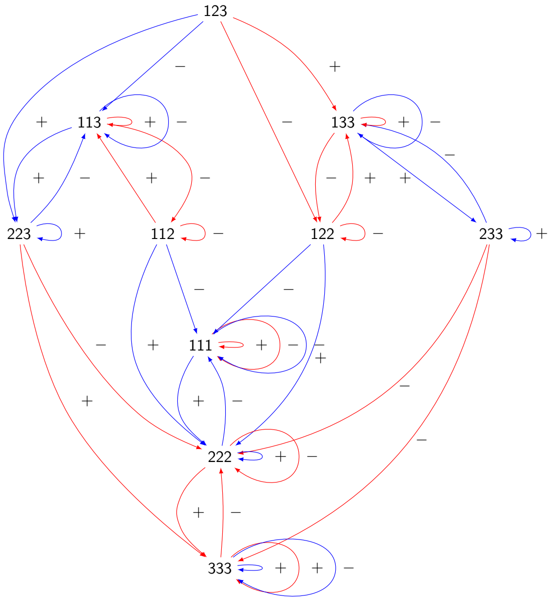
The right Cayley graph of $\mathcal{O}_3$



R-preorder (Green 50)

Definition ($\mathcal{R}$-preorder)

$$x \leq_R y \quad \text{if} \quad x \in yM$$

- $\mathcal{R}$-**class** $\mathcal{R}(x)$: strongly connected component
- $\mathcal{R}$-**order** on $\mathcal{R}$-classes
- $\mathcal{R}$-**trivial monoid**: all $\mathcal{R}$-classes are trivial

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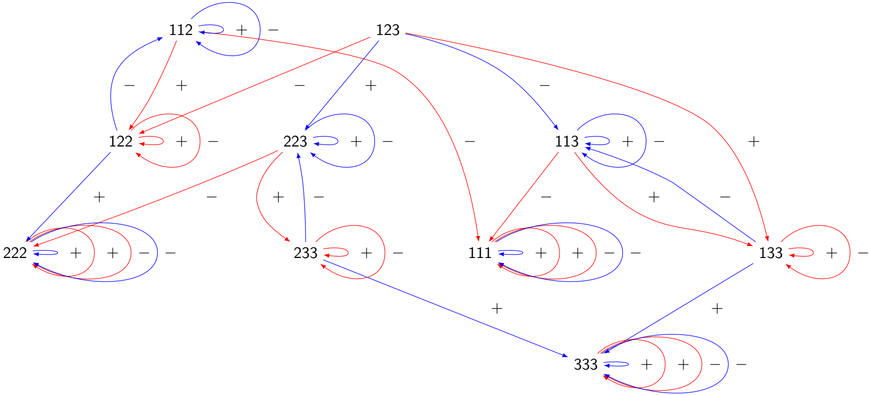
The left Cayley graph of $\mathcal{O}_3$



Left-right Cayley graph, $\mathcal{J}$-preorder

Problem

- *Why do we get several times the same module?*
- *Can we exploit associativity?*

Definition ($\mathcal{J}$-preorder)

$$x \leq_J y \quad \text{if} \quad x \in MyM$$

- **Left-right Cayley graph**
- $\mathcal{J}$**-class**
- $\mathcal{J}$**-preorder**
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Example: the left-right Cayley graph for $\mathcal{O}3$

The eggbox picture

Proposition

Let J be a $\mathcal{J}$-class. Then,

$$J \quad \approx_{M-\text{mod}-M} \quad L \times R$$

where L and R are respectively left and right classes

If e is an idempotent:

$$\mathcal{J}(e) = \mathcal{L}(e)\mathcal{R}(e)$$

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Aperiodic Rees matrix monoid

Definition (Aperiodic Rees matrix monoid)

- Take $M := \{1, \ldots, n\} \times \{1, \ldots, m\}$
- Choose which (i, j) shall be idempotent
  Either none, or at least one per line and per column
- Set

$$(i, j).(j, k) = \begin{cases} (i, k) & \text{if } (k, i) \text{ is idempotent} \\ 0 & \text{otherwise} \end{cases}$$

The structure of M is encoded by the **eggbox matrix** P.

Err, but groups are monoids, aren't they?

- A group has a single $\mathcal{J}$-class!
- So far we have been only speaking about **aperiodic monoids**
- **Rees matrix monoids** $R(P, G)$
- **Schützenberger representations**:
  - $R(e)$ is a G-mod-M module
  - $R(e)$ is a free G-mod (think coset decomposition)
  - Symmetrically for $L(e)$

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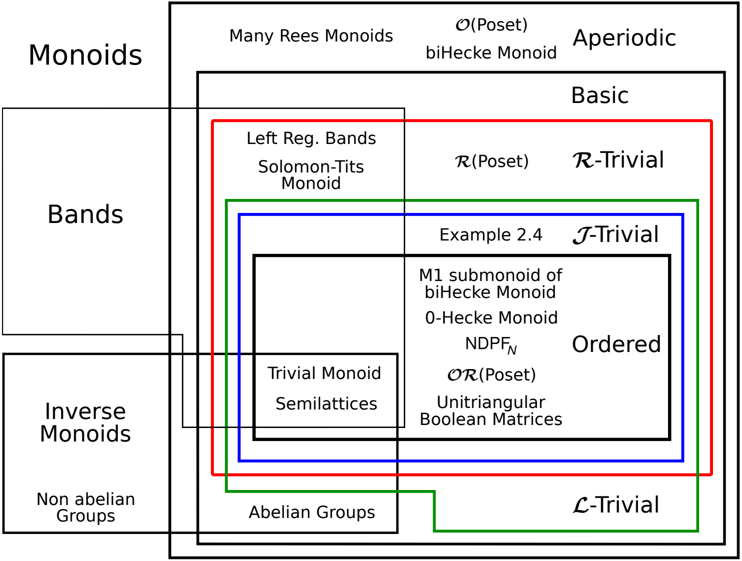
Zoology of monoids



What's known about linear representations?

Finite groups

- Semi-simple: simple = projective (characteristic 0)
- Character theory
- Fast $o(n)$ algorithms

Finite dimensional algebras

- One-to-one correspondance Simple - Projective modules
- Algorithmic: minimal polynomial, linear algebra: $O(n^3)$
- In practice: dimension ≤ 1000

Monoids

- In progress: Clifford, Munn, Ponizovskiĭ, Mc Alister, Putcha, Saliola, Steinberg, Margolis, ...

Simple modules, Clifford, Munn, Ponizovskiĭ

Let M be an aperiodic monoid

Proposition

For R a regular $\mathcal{R}$-class of M,

$$\mathrm{rad}\mathbb{K}R = \{x \in \mathbb{K}R,\ x.r = 0 \forall r \in R\}$$

Equivalently: $\mathrm{rad}\mathbb{K}R$ *is the kernel of the eggbox matrix*

Proposition

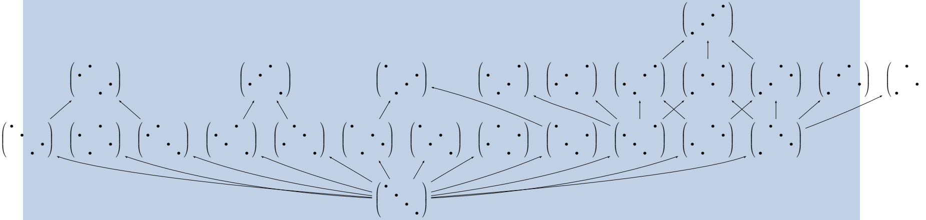
Define $S_i := \mathrm{top}R_i = \mathbb{K}R_i/\mathrm{rad}\mathbb{K}R_i$
All simple modules of M: $(S_i)_{i\in I}$

Simple modules for the biHecke Monoid

First non trivial aperiodic monoid admitting a combinatorial description of the (dimension) of the simple modules [HST09]

Combinatorial ingredients

- Intervals in right order (for right classes)
- Möbius inversion along the cutting poset (for moding out the radical)



Simple modules, Clifford, Munn, Ponizovskiĭ

Proposition

- *Take a J-class and G_i its group*
- *Take a simple module $S_{i,j}^{G_i}$ of G_i.*
- *Define $S_{i,j} := \operatorname{top}(S_{i,j}^{G_i} \uparrow_{G_i}^{M})$*

This construction gives each simple module of M exactly once.

In practice

- $\operatorname{top} R_i$ is only semi-simple
- Decompose it further as G_i-mod

Character table of a monoid, Mc Alister 1972

Definition (Character of an element x acting on a module V)

$\chi_V(x)$: trace of the matrix of x acting on V.

For each J-class, take one element $g_{i,j}$ in each conjugacy class of G_i.

Definition (Character of a module V)

$$\chi_V := \sum \chi_V(g_{i,j}) p_{i,j}$$

where $p_{i,j}$ are formal indeterminates.

Definition (Character table)

$$(\chi_{S_{i,j}}(g_{i',j'}))_{(i,j),(i',j')}$$

Properties of the character table

Theorem

For an aperiodic monoid, the character table is upper unitriangular

For a monoid, the character table is upper block triangular, with the blocks being the character tables of the groups G_i

Corollary

Characters can be used to compute composition factors!

Cartan invariant matrix as linear refinement of $\mathcal{J}$-preorder

Definition

A: finite dimensional algebra (e. g. $A = \mathbb{Q}[M]$)
A is an A-mod-A module (or $A^{\mathrm{op}} \otimes A$-module)
Composition series: $\{0\} = A_0 \subset \cdots \subset A_\ell = A$

Proposition

$$A_{k+1}/A_k \approx_{A-\mathrm{mod}-A} S_i \otimes S_j^*$$

where S_i is a simple left module and S_j^* is a simple right module

See e.g. Curtis-Reiner.

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The Cartan (invariants) matrix

Definition

$C = (c_{i,j})_{i,j}$, with:

$$c_{i,j} \quad = \quad |\{k,\ A_{k+1}/A_k\ \approx_{A-\mathrm{mod}-A}\ S_i \otimes S_j\}|$$

Equivalent definitions:

- On the left: $[P_j] = \sum_i c_{i,j}[S_i]$
- On the right: $[P_i] = \sum_j c_{i,j}[S_j]$
- Dimension of sandwiches by idempotents: $c_{i,j} = \dim e_i A e_j$

Cartan matrix by orthogonal idempotents

1. Build a decomposition of the identity into orthogonal idempotents e_i
2. Compute $e_i A e_j$
3. Build the projective modules as $e_i A$

Problem

Non trivial construction!

- *0-Hecke in type A: combinatorial formula [Denton'10]*
- *$\mathcal{R}$-trivial: recursive formula [Berg,Bergeron,Bhargava,Saliola'10]*
- *Aperiodic?*
- *Algebra: may require arbitrary algebraic extensions*

Idempotent free approach?

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Idempotent free approach?

Cartan matrix of a monoid by characters

Let M be a finite monoid
$(g_i)_{i\in I}$: representatives of the conjugacy classes
Define the matrix $\overline{C} = (\overline{c}_{i,j})_{i,j\in I}$ by

$$\overline{c}_{i,j} := \left|\{s \in M, g_i s g_j^* = s\}\right| .$$

Lemma (T'11)

The bi-character of $\mathbb{K}M$ is given by

$$\chi_{\mathbb{K}M} = \sum_{i,j\in I} \overline{c}_{i,j} p_i \otimes p_j^* .$$

Cartan matrix for a general monoid

Algorithm to compute the Cartan matrix

- Build the simple modules (linear algebra)
- Compute the character table, i.e. the change of basis $p_i \mapsto s_i$
- Compute the matrix C (purely combinatorial)
- Apply the change of bases $p_i \otimes p_j^* \mapsto s_i \otimes s_j^*$

Cartan matrix for aperiodic monoids

Remark

- *The composition series of* $\mathbb{Q}[M]$ *refines the decomposition of* M *into* $\mathcal{J}$*-classes*
- *For* J *a* $\mathcal{J}$*-class of the form* $L \times R$*:*

$$\mathbb{Q}J \quad \approx_{\mathbb{Q}[M]-\mathrm{mod}-\mathbb{Q}[M]} \quad \mathbb{Q}L \otimes \mathbb{Q}R$$

Proposition (T'11)

M_L*: decomposition matrix of left class modules into simples*
M_R*: decomposition matrix of right class modules into simples*
Then, $C = M_{\mathcal{L}}^t M_{\mathcal{R}}$

Remark: $M_{\mathcal{L}}$ and $M_{\mathcal{R}}$ are upper unitriangular on regular $\mathcal{J}$-classes

Cartan matrix of aperiodic monoids

Algorithm [T'11]

Input: an aperiodic monoid

1. Construct representatives of left and right class modules
2. Construct the simple modules as quotients thereof
3. Compute the character table
4. Compute the character of each left and right class module
5. Compute the decomposition matrices $M_{\mathcal{L}}$ and $M_{\mathcal{R}}$

Output: The cartan matrix $C = M_{\mathcal{L}}^t M_{\mathcal{R}}$

Advantages of this algorithm

- Splits the linear algebra in small chunks
- Take advantage of the redundancy
- Rough complexity: $O(\sum_{i\in I} |R_i|^3)$
- Cartan matrix of a monoid of size 31103 in one hour
- Potential for parallelism!

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Theoretical consequences

- Mostly characteristic free
- No algebraic extension needed (no surprise)
- Generalization to PIDs ($\mathbb{Z}$, ...)
- Completely combinatorial for $\mathcal{J}$-trivial monoids [Denton, Hivert, Schilling, T'2010]

Problems

- *Quiver?*
- *Socle/Radical filtration?*
- *Construction of projective modules*
- *Interesting examples in combinatorics?*

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Cartan matrix of monoids in characteristic zero

Same principle, with some complications

Remark

For a $\mathcal{J}$-class J:

$$\mathbb{K}J \quad \approx_{\mathbb{K}M-\text{mod}-\mathbb{K}M} \quad \mathbb{K}L \otimes_{\mathbb{K}G} \mathbb{K}R$$

Algorithm [T'11]

- For each $\mathcal{J}$-class J, regular or not
  - Compute the group G
  - Compute the M-mod-G character of $\mathbb{K}L$
  - Compute the G-mod-M character of $\mathbb{K}R$
  - Recombine the information to get the M-mod-M character of $\mathbb{K}J$
- Collect all and change basis