Sorting monoids and algebras on Coxeter groups

Florent Hivert[1] Anne Schilling[2] Nicolas M. Thiéry[2,3]

[1]LITIS/LIFAR, Université Rouen, France

[2]University of California at Davis, USA

[3]Laboratoire de Mathématiques d'Orsay, Université Paris Sud, France

New Orleans, March 20th of 2009

arXiv:0711.1561v1 [math.RT]
arXiv:0804.3781v1 [math.RT]
+ research in progress

Orders on words and on Coxeter group elements

Definition (Orders on words)

Let $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$:

- u **left factor** of v if $v = u_1 \cdots u_k \cdots$
- u **right factor** of v if $v = \cdots u_1 \cdots u_k$
- u **factor** of v if $v = \cdots u_1 \cdots u_k \cdots$
- u **subword** of v if $v = \cdots u_1 \cdots u_2 \cdots u_k \cdots$

Definition (Orders on Coxeter group elements)

- Right weak order
- Left weak order
- Left-right weak order
- Bruhat order

Orders on words and on Coxeter group elements

Definition (Orders on words)

Let $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$:

- u **left factor** of v if $v = u_1 \cdots u_k \cdots$
- u **right factor** of v if $v = \cdots u_1 \cdots u_k$
- u **factor** of v if $v = \cdots u_1 \cdots u_k \cdots$
- u **subword** of v if $v = \cdots u_1 \cdots u_2 \cdots u_k \cdots$

Definition (Orders on Coxeter group elements)

- Right weak order
- Left weak order
- Left-right weak order
- Bruhat order

Orders on words and on Coxeter group elements

Definition (Orders on words)

Let $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$:

- u **left factor** of v if $v = u_1 \cdots u_k \cdots$
- u **right factor** of v if $v = \cdots u_1 \cdots u_k$
- u **factor** of v if $v = \cdots u_1 \cdots u_k \cdots$
- u **subword** of v if $v = \cdots u_1 \cdots u_2 \cdots u_k \cdots$

Definition (Orders on Coxeter group elements)

- Right weak order
- Left weak order
- Left-right weak order
- Bruhat order

Orders on words and on Coxeter group elements

Definition (Orders on words)

Let $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$:

- u **left factor** of v if $v = u_1 \cdots u_k \cdots$
- u **right factor** of v if $v = \cdots u_1 \cdots u_k$
- u **factor** of v if $v = \cdots u_1 \cdots u_k \cdots$
- u **subword** of v if $v = \cdots u_1 \cdots u_2 \cdots u_k \cdots$

Definition (Orders on Coxeter group elements)

- Right weak order
- Left weak order
- Left-right weak order
- Bruhat order

The bi-Hecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \dots, s_1, s_2, \dots \rangle$
$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Conjecture

$M(W)$ *admits* $|W|$ *simple / indecomposable projective modules*

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w . \dim P_w$$

The bi-Hecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \ldots, s_1, s_2, \ldots \rangle$
$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Conjecture

$M(W)$ admits $|W|$ simple / indecomposable projective modules

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w . \dim P_w$$

The bi-Hecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \ldots, s_1, s_2, \ldots \rangle$
$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Conjecture

$M(W)$ admits $|W|$ simple / indecomposable projective modules

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w . \dim P_w$$

The bi-Hecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \ldots, s_1, s_2, \ldots \rangle$
$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Conjecture

$M(W)$ admits $|W|$ simple / indecomposable projective modules

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w . \dim P_w$$

The bi-Hecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \ldots, s_1, s_2, \ldots \rangle$
$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Conjecture

$M(W)$ *admits* $|W|$ *simple / indecomposable projective modules*

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w . \dim P_w$$

The bi-Hecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \dots, s_1, s_2, \dots \rangle$
$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Conjecture

$M(W)$ *admits* $|W|$ *simple / indecomposable projective modules*

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w . \dim P_w$$

Representation theory of algebras

Module: vector space V with a morphism $M \mapsto \mathrm{End}(V)$
Simple module: V contains no nontrivial submodule
Indecomposable module: V cannot be written as $V = V_1 \oplus V_2$
Projective module: $V \oplus \cdots = \mathbb{C}[M] \oplus \cdots \oplus \mathbb{C}[M]$

Theorem (See e.g. Curtis-Reiner)

Simple modules $\leftrightarrow$ indecomposable projective modules
Dimension formula, ...

Keys role of idempotents:

- eM projective module
- If $f = uev$ then fM is isomorphic to a submodule of eM

Representation theory of algebras

Module: vector space V with a morphism $M \mapsto \mathrm{End}(V)$
Simple module: V contains no nontrivial submodule
Indecomposable module: V cannot be written as $V = V_1 \oplus V_2$
Projective module: $V \oplus \cdots = \mathbb{C}[M] \oplus \cdots \oplus \mathbb{C}[M]$

Theorem (See e.g. Curtis-Reiner)

Simple modules $\leftrightarrow$ indecomposable projective modules
Dimension formula, ...

Keys role of idempotents:

- eM projective module
- If $f = uev$ then fM is isomorphic to a submodule of eM

Representation theory of algebras

Module: vector space V with a morphism $M \mapsto \mathrm{End}(V)$
Simple module: V contains no nontrivial submodule
Indecomposable module: V cannot be written as $V = V_1 \oplus V_2$
Projective module: $V \oplus \cdots = \mathbb{C}[M] \oplus \cdots \oplus \mathbb{C}[M]$

Theorem (See e.g. Curtis-Reiner)

Simple modules $\leftrightarrow$ indecomposable projective modules
Dimension formula, ...

Keys role of idempotents:

- eM projective module
- If $f = uev$ then fM is isomorphic to a submodule of eM

Representation theory of monoids

Definition (J-class)

$x, y \in M$ are in the same J-class iff

$$x = uyv \text{ and } y = u'xv' \text{for some} u, v, u', v' \in M$$

A J-class is regular iff it contains an idempotent

Theorem (See e.g. Ganyushkin, Mazorchuk, Steinberg 07)

The regular J-classes (essentially) determine the simple modules.

Representation theory of monoids

Definition (J-class)

$x, y \in M$ are in the same J-class iff

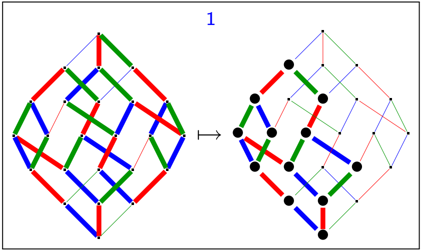
$$x = uyv \text{ and } y = u'xv' \text{for some} u, v, u', v' \in M$$

A J-class is regular iff it contains an idempotent

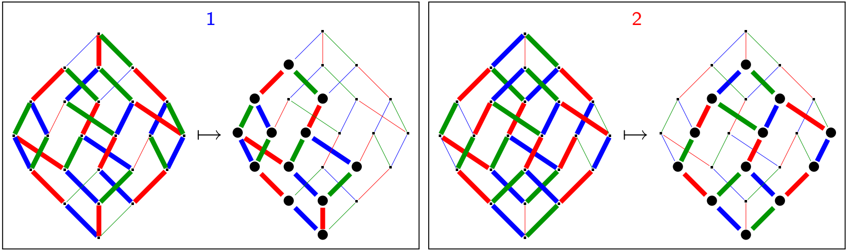
Theorem (See e.g. Ganyushkin, Mazorchuk, Steinberg 07)

The regular J-classes (essentially) determine the simple modules.

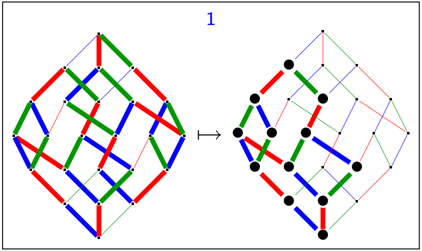
Some elements of the monoid

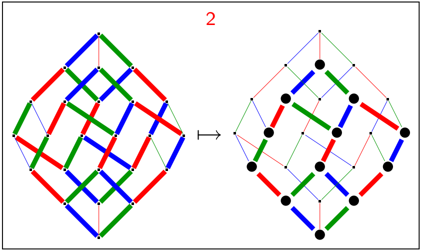


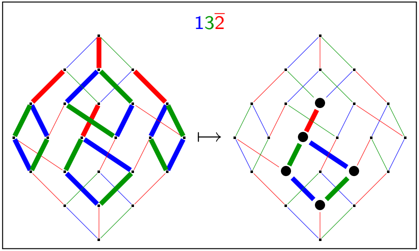
Some elements of the monoid



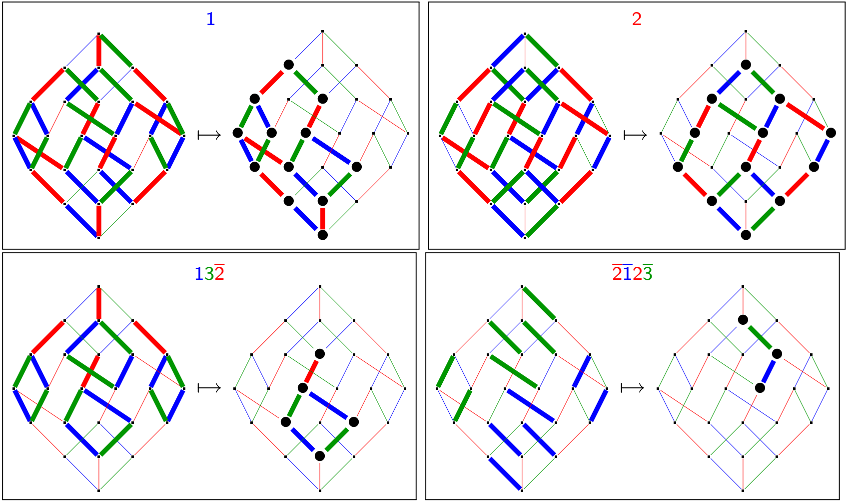
Some elements of the monoid







Some elements of the monoid



Key combinatorial lemma

Lemma

For $f \in M$, and $v \in W$:

- *Either $(s_i v).f = s_i\,(v.f)$ or $(s_i v).f = v.f$*

Corollary

- *If $w = uv$, then $(uv).f = u'\,(v.f)$, where $u' \leq_{Bruhat} u$*
- *Preservation of left order: $u \leq_L v \Longrightarrow u.f \leq_L v.f$*
- *f in M is determined by its fibers and image set*
- *When $u' = u$, f is an isomorphism on $[u, v]_L$*

Key combinatorial lemma

Lemma

For $f \in M$, and $v \in W$:

- *Either* $(s_i v).f = s_i\ (v.f)$ *or* $(s_i v).f = v.f$

Corollary

- *If* $w = uv$, *then* $(uv).f = u'\ (v.f)$, *where* $u' <_{Bruhat} u$
- *Preservation of left order:* $u \leq_L v \Longrightarrow u.f \leq_L v.f$
- f *in* M *is determined by its fibers and image set*
- *When* $u' = u$, f *is an isomorphism on* $[u, v]_L$

Key combinatorial lemma

Lemma

For $f \in M$, and $v \in W$:

- *Either $(s_i v).f = s_i\ (v.f)$ or $(s_i v).f = v.f$*

Corollary

- *If $w = uv$, then $(uv).f = u'\ (v.f)$, where $u' <_{Bruhat} u$*
- *Preservation of left order: $u \leq_L v \Longrightarrow u.f \leq_L v.f$*
- *f in M is determined by its fibers and image set*
- *When $u' = u$, f is an isomorphism on $[u, v]_L$*

Key combinatorial lemma

Lemma

For $f \in M$, and $v \in W$:

- *Either $(s_i v).f = s_i\ (v.f)$ or $(s_i v).f = v.f$*

Corollary

- *If $w = uv$, then $(uv).f = u'\ (v.f)$, where $u' <_{Bruhat} u$*
- *Preservation of left order: $u \leq_L v \Longrightarrow u.f \leq_L v.f$*
- *f in M is determined by its fibers and image set*
- *When $u' = u$, f is an isomorphism on $[u, v]_L$*

Key combinatorial lemma

Lemma

For $f \in M$, and $v \in W$:

- *Either* $(s_i v).f = s_i\ (v.f)$ *or* $(s_i v).f = v.f$

Corollary

- *If* $w = uv$, *then* $(uv).f = u'\ (v.f)$, *where* $u' <_{Bruhat} u$
- *Preservation of left order:* $u \leq_L v \Longrightarrow u.f \leq_L v.f$
- f *in* M *is determined by its fibers and image set*
- *When* $u' = u$, f *is an isomorphism on* $[u, v]_L$

Key combinatorial lemma

Lemma

For $f \in M$, and $v \in W$:

- *Either $(s_i v).f = s_i\ (v.f)$ or $(s_i v).f = v.f$*

Corollary

- *If $w = uv$, then $(uv).f = u'\ (v.f)$, where $u' <_{Bruhat} u$*
- *Preservation of left order: $u \leq_L v \Longrightarrow u.f \leq_L v.f$*
- *f in M is determined by its fibers and image set*
- *When $u' = u$, f is an isomorphism on $[u, v]_L$*

Main theorem

Theorem

$M(W)$ admits $|W|$ simple modules

Proof.

- M acts transitively on intervals $[u, v]_L$
- The image set of an idempotent is an interval
- Existence of an idempotent e_w with image set $[1, w]_L$ for each $w \in W$
- The e_w form a transversal of the regular J-classes
  - $f = uev$ if and only if $\operatorname{im} f$ is a sub interval of $\operatorname{im} e$
- The groups $e_w M e_w$ are trivial

□

Main theorem

Theorem

$M(W)$ admits $|W|$ simple modules

Proof.

- M acts transitively on intervals $[u, v]_L$
- The image set of an idempotent is an interval
- Existence of an idempotent e_w with image set $[1, w]_L$ for each $w \in W$
- The e_w form a transversal of the regular J-classes
  - $f = uev$ if and only if $\operatorname{im} f$ is a sub interval of $\operatorname{im} e$
- The groups $e_w M e_w$ are trivial

□

Main theorem

Theorem

$M(W)$ admits $|W|$ simple modules

Proof.

- M acts transitively on intervals $[u, v]_L$
- The image set of an idempotent is an interval
- Existence of an idempotent e_w with image set $[1, w]_L$ for each $w \in W$
- The e_w form a transversal of the regular J-classes
  - $f = uev$ if and only if $\mathrm{im} f$ is a sub interval of $\mathrm{im} e$
- The groups $e_w M e_w$ are trivial

□

Main theorem

Theorem

$M(W)$ admits $|W|$ simple modules

Proof.

- M acts transitively on intervals $[u, v]_L$
- The image set of an idempotent is an interval
- Existence of an idempotent e_w with image set $[1, w]_L$ for each $w \in W$
- The e_w form a transversal of the regular J-classes
  - $f = uev$ if and only if $\operatorname{im} f$ is a sub interval of $\operatorname{im} e$
- The groups $e_w M e_w$ are trivial

□

Main theorem

Theorem

$M(W)$ admits $|W|$ simple modules

Proof.

- M acts transitively on intervals $[u, v]_L$
- The image set of an idempotent is an interval
- Existence of an idempotent e_w with image set $[1, w]_L$ for each $w \in W$
- The e_w form a transversal of the regular J-classes
  - $f = uev$ if and only if $\mathrm{im} f$ is a sub interval of $\mathrm{im} e$
- The groups $e_w M e_w$ are trivial

□

Main theorem

Theorem

$M(W)$ admits $|W|$ simple modules

Proof.

- M acts transitively on intervals $[u, v]_L$
- The image set of an idempotent is an interval
- Existence of an idempotent e_w with image set $[1, w]_L$ for each $w \in W$
- The e_w form a transversal of the regular J-classes
  - $f = uev$ if and only if $\mathrm{im} f$ is a sub interval of $\mathrm{im} e$
- The groups $e_w M e_w$ are trivial

□

Dimension of simple and projective modules?

Submonoid: $M_1 = \{f \in M, f(1) = 1\}$
Properties:

- Idempotents: $(e_w)_{w \in W}$
- $|W|$ simple modules of dimension 1
- Semi simple quotient: monoid algebra of $(W, \vee_L)$
- Conjugacy order among idempotents: $<_{LR}$
- $\dim P_w = |\{f \in M_1, f(w) = w ...\}|$?

Inducing those results to M?

Hecke group algebras

A silly idea during a brainstorm (Thibon, Novelli, H., T., 2003)

Definition (Hecke group algebra HW of a Coxeter group W)

Glue $\mathbb{C}[W]$ and H$(W)(0)$ on their right regular representations:

$$HW := \mathbb{Q}[\pi_1, \pi_2, \dots, s_1, s_2, \dots] \quad \subset \quad \mathrm{End}(\mathbb{C}W)$$

- Any interesting structure?
- Contains all Hecke algebras by construction
- Type A: dimension and dimension of the radical in the Sloane!

Hecke group algebras

A silly idea during a brainstorm (Thibon, Novelli, H., T., 2003)

Definition (Hecke group algebra HW of a Coxeter group W)

Glue $\mathbb{C}[W]$ and H$(W)(0)$ on their right regular representations:

$$HW := \mathbb{Q}[\pi_1, \pi_2, \ldots, s_1, s_2, \ldots] \quad \subset \quad \mathrm{End}(\mathbb{C}W)$$

- Any interesting structure?
- Contains all Hecke algebras by construction
- Type A: dimension and dimension of the radical in the Sloane!

Hecke group algebras

A silly idea during a brainstorm (Thibon, Novelli, H., T., 2003)

Definition (Hecke group algebra HW of a Coxeter group W)

Glue $\mathbb{C}[W]$ and H$(W)(0)$ on their right regular representations:

$$HW := \mathbb{Q}[\pi_1, \pi_2, \ldots, s_1, s_2, \ldots] \quad \subset \quad \mathrm{End}(\mathbb{C}W)$$

- Any interesting structure?
- Contains all Hecke algebras by construction
- Type A: dimension and dimension of the radical in the Sloane!

Hecke group algebras

A silly idea during a brainstorm (Thibon, Novelli, H., T., 2003)

Definition (Hecke group algebra HW of a Coxeter group W)

Glue $\mathbb{C}[W]$ and H$(W)(0)$ on their right regular representations:

$$HW := \mathbb{Q}[\pi_1, \pi_2, \dots, s_1, s_2, \dots] \quad \subset \quad \mathrm{End}(\mathbb{C}W)$$

- Any interesting structure?
- Contains all Hecke algebras by construction
- Type A: dimension and dimension of the radical in the Sloane!

The Hecke group algebra of rank 1

$$W := \{1, s\} \qquad \mathbb{C}W := \mathbb{C}.1 \oplus \mathbb{C}.s$$

Basis

$$\left\{ \quad \mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \pi = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \right\}$$

Relations

$$s\pi = \pi, \qquad \pi s = \overline{\pi}, \qquad \overline{\pi} + \pi = 1 + s$$

Dimension 1 simple and projective module

$$(1-s).\mathrm{id} = (1-s), \qquad (1-s).s = -(1-s), \qquad (1-s).\pi = 0$$

The Hecke group algebra of rank 1

$$W := \{1, s\} \qquad \mathbb{C}W := \mathbb{C}.1 \oplus \mathbb{C}.s$$

Basis

$$\left\{ \quad \mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \pi = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \right\}$$

Relations

$$s\pi = \pi, \qquad \pi s = \overline{\pi}, \qquad \overline{\pi} + \pi = 1 + s$$

Dimension 1 simple and projective module

$$(1-s).\mathrm{id} = (1-s), \qquad (1-s).s = -(1-s), \qquad (1-s).\pi = 0$$

The Hecke group algebra of rank 1

$$W := \{1, s\} \qquad \mathbb{C}W := \mathbb{C}.1 \oplus \mathbb{C}.s$$

Basis

$$\left\{ \quad \mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \pi = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \right\}$$

Relations

$$s\pi = \pi, \qquad \pi s = \overline{\pi}, \qquad \overline{\pi} + \pi = 1 + s$$

Dimension 1 simple and projective module

$$(1-s).\mathrm{id} = (1-s), \qquad (1-s).s = -(1-s), \qquad (1-s).\pi = 0$$

Structure theory of Hecke group algebras (I)

Definition

Left / right descent sets:

$$\mathrm{D}_R(w) := \{i \in S \mid w = w' s_i\} \qquad \mathrm{D}_L(w) := \{i \in S \mid w = s_i w'\}$$

$v \in \mathbb{C}W$ ***i*-left antisymmetric** if $s_i v = -v$

Theorem (H.,T., 2005)

Basis of $\mathrm{H}W$: $\{w\pi_{w'} \mid \mathrm{D}_R(w) \cap \mathrm{D}_L(w') = \emptyset\}$

$\mathrm{H}W$ *algebra of left antisymmetry preserving operators*

$\mathrm{H}W^*$ *algebra of left symmetry preserving operators*

- Proof: simultaneous triangularity of basis and linear relations
- Straightening relations? Presentation?

Structure theory of Hecke group algebras (I)

Definition

Left / right descent sets:

$$\mathrm{D}_R(w) := \{i \in S \mid w = w' s_i\} \qquad \mathrm{D}_L(w) := \{i \in S \mid w = s_i w'\}$$

$v \in \mathbb{C}W$ ***i*-left antisymmetric** if $s_i v = -v$

Theorem (H.,T., 2005)

Basis of $\mathrm{H}W$*:* $\{w\pi_{w'} \mid \mathrm{D}_R(w) \cap \mathrm{D}_L(w') = \emptyset\}$

$\mathrm{H}W$ *algebra of left antisymmetry preserving operators*

$\mathrm{H}W^*$ *algebra of left symmetry preserving operators*

- Proof: simultaneous triangularity of basis and linear relations
- Straightening relations? Presentation?

Structure theory of Hecke group algebras (I)

Definition

Left / right descent sets:

$$\mathrm{D}_R(w) := \{i \in S \mid w = w' s_i\} \qquad \mathrm{D}_L(w) := \{i \in S \mid w = s_i w'\}$$

$v \in \mathbb{C}W$ i-**left antisymmetric** if $s_i v = -v$

Theorem (H.,T., 2005)

Basis of $\mathrm{H}W$*:* $\{w\pi_{w'} \mid \mathrm{D}_R(w) \cap \mathrm{D}_L(w') = \emptyset\}$

$\mathrm{H}W$ *algebra of left antisymmetry preserving operators*

$\mathrm{H}W^*$ *algebra of left symmetry preserving operators*

- Proof: simultaneous triangularity of basis and linear relations
- Straightening relations? Presentation?

Structure theory of Hecke group algebras (I)

Definition

Left / right descent sets:

$$\mathrm{D}_R(w) := \{i \in S \mid w = w' s_i\} \qquad \mathrm{D}_L(w) := \{i \in S \mid w = s_i w'\}$$

$v \in \mathbb{C}W$ ***i*-left antisymmetric** if $s_i v = -v$

Theorem (H.,T., 2005)

Basis of $\mathrm{H}W$*:* $\{w\pi_{w'} \mid \mathrm{D}_R(w) \cap \mathrm{D}_L(w') = \emptyset\}$

$\mathrm{H}W$ *algebra of left antisymmetry preserving operators*

$\mathrm{H}W^*$ *algebra of left symmetry preserving operators*

- Proof: simultaneous triangularity of basis and linear relations
- Straightening relations? Presentation?

Structure theory of Hecke group algebras (II)

- Modules P_I for $I \subset S$:

$$P_I := \{v \in \mathbb{C}W \mid s_i v = -s_i, \forall i \in I\}$$

- Boolean lattice: $I \subset J \Longrightarrow P_J \subset P_I$
- Combinatorics of descent classes: $\dim P_I = |{}^I_S W|$
- **Good basis** of $\mathbb{C}W$: compatible with restriction on each P_I:
  - Par ex: $\left\{ v_w := \sum_{w' \in W_{S \setminus D_L(w)}} (-1)^{l(w')} w'w \;\middle|\; w \in W \right\}$
  - Or the Kazhdan-Lusztig basis to be fancy

Proposition

Realization of the Hecke group algebra as digraph algebra:
Basis: $\{e_{w,w'} \mid D_L(w) \subset D_L(w')\}$, *where* $e_{w,w'}(v_{w''})\delta_{w'',w} v_{w'}$

Structure theory of Hecke group algebras (II)

- Modules P_I for $I \subset S$:

$$P_I := \{v \in \mathbb{C}W \mid s_i v = -s_i, \forall i \in I\}$$

- Boolean lattice: $I \subset J \Longrightarrow P_J \subset P_I$
- Combinatorics of descent classes: $\dim P_I = |{}^I_S W|$
- **Good basis** of $\mathbb{C}W$: compatible with restriction on each P_I:
  - Par ex: $\left\{ v_w := \sum_{w' \in W_{S \setminus D_L(w)}} (-1)^{l(w')} w'w \;\middle|\; w \in W \right\}$
  - Or the Kazhdan-Lusztig basis to be fancy

Proposition

Realization of the Hecke group algebra as digraph algebra:
Basis: $\{e_{w,w'} \mid D_L(w) \subset D_L(w')\}$, *where* $e_{w,w'}(v_{w''})\delta_{w'',w} v_{w'}$

Structure theory of Hecke group algebras (II)

- Modules P_I for $I \subset S$:

$$P_I := \{v \in \mathbb{C}W \mid s_i v = -s_i, \forall i \in I\}$$

- Boolean lattice: $I \subset J \Longrightarrow P_J \subset P_I$
- Combinatorics of descent classes: $\dim P_I = |{}^I_S W|$
- **Good basis** of $\mathbb{C}W$: compatible with restriction on each P_I:
  - Par ex: $\left\{ v_w := \sum_{w' \in W_{S \setminus \mathrm{D}_L(w)}} (-1)^{l(w')} w' w \;\middle|\; w \in W \right\}$
  - Or the Kazhdan-Lusztig basis to be fancy

Proposition

Realization of the Hecke group algebra as digraph algebra:
Basis: $\{e_{w,w'} \mid \mathrm{D}_L(w) \subset \mathrm{D}_L(w')\}$, *where* $e_{w,w'}(v_{w''}) \delta_{w'',w} v_{w'}$

Structure theory of Hecke group algebras (II)

- Modules P_I for $I \subset S$:

$$P_I := \{v \in \mathbb{C}W \mid s_i v = -s_i, \forall i \in I\}$$

- Boolean lattice: $I \subset J \Longrightarrow P_J \subset P_I$
- Combinatorics of descent classes: $\dim P_I = |{}^I_S W|$
- **Good basis** of $\mathbb{C}W$: compatible with restriction on each P_I:
  - Par ex: $\left\{ v_w := \sum_{w' \in W_{S \setminus \mathrm{D}_L(w)}} (-1)^{l(w')} w'w \quad \middle| \quad w \in W \right\}$
  - Or the Kazhdan-Lusztig basis to be fancy

Proposition

Realization of the Hecke group algebra as digraph algebra:
Basis: $\{e_{w,w'} \mid \mathrm{D}_L(w) \subset \mathrm{D}_L(w')\}$, *where* $e_{w,w'}(v_{w''})\delta_{w'',w} v_{w'}$

Structure theory of Hecke group algebras (II)

- Modules P_I for $I \subset S$:

$$P_I := \{v \in \mathbb{C}W \mid s_i v = -s_i, \forall i \in I\}$$

- Boolean lattice: $I \subset J \Longrightarrow P_J \subset P_I$
- Combinatorics of descent classes: $\dim P_I = |{}^I_S W|$
- **Good basis** of $\mathbb{C}W$: compatible with restriction on each P_I:
  - Par ex: $\left\{ v_w := \sum_{w' \in W_{S \setminus \mathrm{D}_L(w)}} (-1)^{l(w')} w' w \quad \middle| \quad w \in W \right\}$
  - Or the Kazhdan-Lusztig basis to be fancy

Proposition

Realization of the Hecke group algebra as digraph algebra:
Basis: $\{e_{w,w'} \mid \mathrm{D}_L(w) \subset \mathrm{D}_L(w')\}$, *where* $e_{w,w'}(v_{w''})\delta_{w'',w} v_{w'}$

Representation theory of Hecke group algebras

Theorem (H.,T., 2005)

- $e_{w,w}$: *max. decomposition of* id *into orthogonal idempotents*
- HW *Morita equivalent to the poset algebra of boolean lattice*
- *Projective modules:* P_I
- *Simple modules:* $S_I := P_I \;/\; \sum_{J \supset I} P_J$

  Left-antisymmetries on I*, left-symmetries on the complement*
  By restriction:
  - *Exactly the Young's ribbon representation of* W
  - *Exactly the projective modules of* H$(W)(0)$

Question

Is there a link with the affine Hecke algebra?

Representation theory of Hecke group algebras

Theorem (H.,T., 2005)

- $e_{w,w}$: *max. decomposition of* id *into orthogonal idempotents*
- HW *Morita equivalent to the poset algebra of boolean lattice*
- *Projective modules:* P_I
- *Simple modules:* $S_I := P_I \;/\; \sum_{J \supset I} P_J$

  Left-antisymmetries on I*, left-symmetries on the complement*
  By restriction:
  - *Exactly the Young's ribbon representation of* W
  - *Exactly the projective modules of* H$(W)(0)$

Question

Is there a link with the affine Hecke algebra?

Representation theory of Hecke group algebras

Theorem (H.,T., 2005)

- *$e_{w,w}$: max. decomposition of* id *into orthogonal idempotents*
- HW *Morita equivalent to the poset algebra of boolean lattice*
- *Projective modules: P_I*
- *Simple modules:* $S_I := P_I \; / \; \sum_{J \supset I} P_J$

  Left-antisymmetries on I, left-symmetries on the complement
  By restriction:
  - *Exactly the Young's ribbon representation of W*
  - *Exactly the projective modules of* H$(W)(0)$

Question

Is there a link with the affine Hecke algebra?

Representation theory of Hecke group algebras

Theorem (H.,T., 2005)

- $e_{w,w}$: *max. decomposition of* id *into orthogonal idempotents*
- $\mathrm{H}W$ *Morita equivalent to the poset algebra of boolean lattice*
- *Projective modules:* P_I
- *Simple modules:* $S_I := P_I \ / \ \sum_{J \supset I} P_J$

  Left-antisymmetries on I*, left-symmetries on the complement*
  By restriction:
  - *Exactly the Young's ribbon representation of* W
  - *Exactly the projective modules of* $\mathrm{H}(W)(0)$

Question

Is there a link with the affine Hecke algebra?

Representation theory of Hecke group algebras

Theorem (H.,T., 2005)

- $e_{w,w}$: *max. decomposition of* id *into orthogonal idempotents*
- HW *Morita equivalent to the poset algebra of boolean lattice*
- *Projective modules:* P_I
- *Simple modules:* $S_I := P_I \;/\; \sum_{J \supset I} P_J$

  Left-antisymmetries on I, *left-symmetries on the complement*
  By restriction:
  - *Exactly the Young's ribbon representation of* W
  - *Exactly the projective modules of* H$(W)(0)$

Question

Is there a link with the affine Hecke algebra?

Representation theory of Hecke group algebras

Theorem (H.,T., 2005)

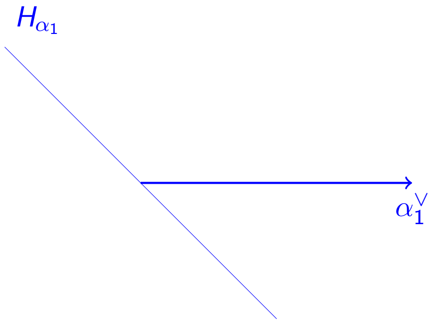
- $e_{w,w}$: *max. decomposition of* id *into orthogonal idempotents*
- HW *Morita equivalent to the poset algebra of boolean lattice*
- *Projective modules:* P_I
- *Simple modules:* $S_I := P_I \;/\; \sum_{J \supset I} P_J$

  Left-antisymmetries on I*, left-symmetries on the complement*
  By restriction:
  - *Exactly the Young's ribbon representation of* W
  - *Exactly the projective modules of* H$(W)(0)$

Question

Is there a link with the affine Hecke algebra?

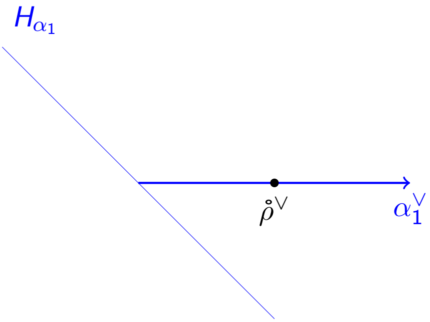
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on* $\mathring{W}$
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$, *not* $\mathrm{H}(\mathring{W})(0)$*!*

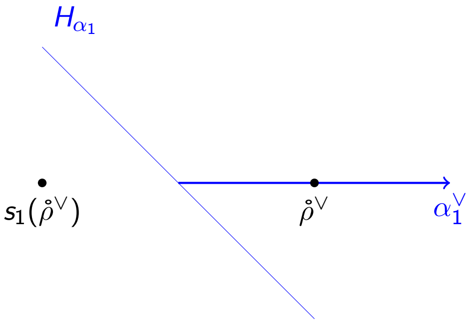
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on* $\mathring{W}$
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$*, not* $\mathrm{H}(\mathring{W})(0)$*!*

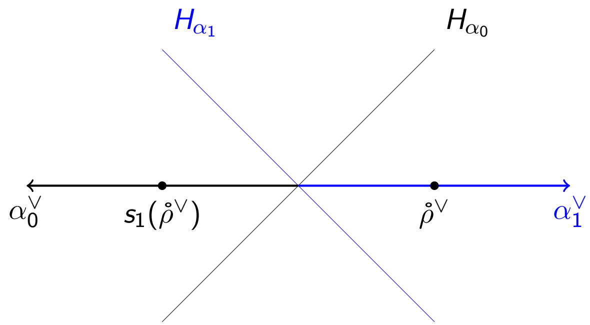
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on* $\mathring{W}$
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$*, not* $\mathrm{H}(\mathring{W})(0)$*!*

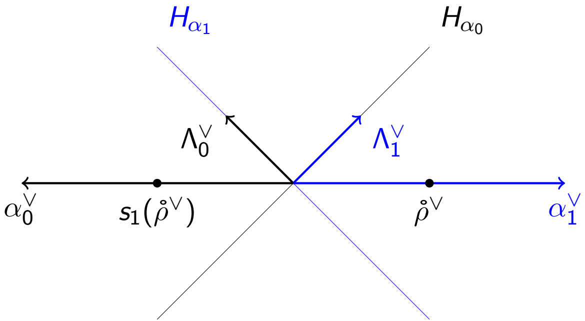
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on $\mathring{W}$*
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$, *not* $\mathrm{H}(\mathring{W})(0)$*!*

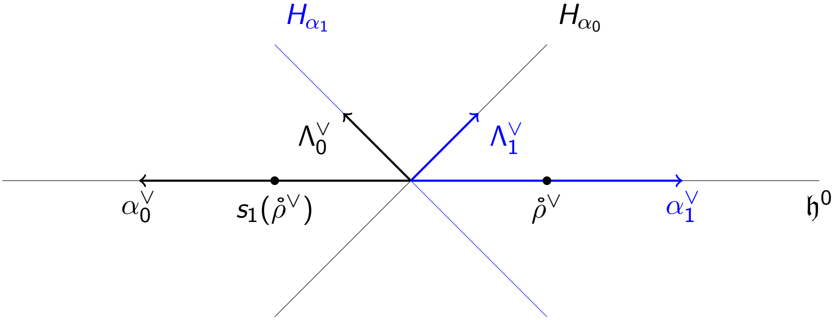
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on* $\mathring{W}$
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$, *not* $\mathrm{H}(\mathring{W})(0)$*!*

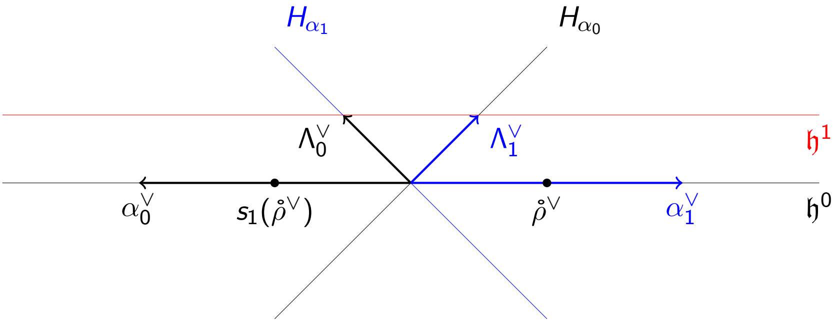
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on* $\mathring{W}$
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$*, not* $\mathrm{H}(\mathring{W})(0)$*!*

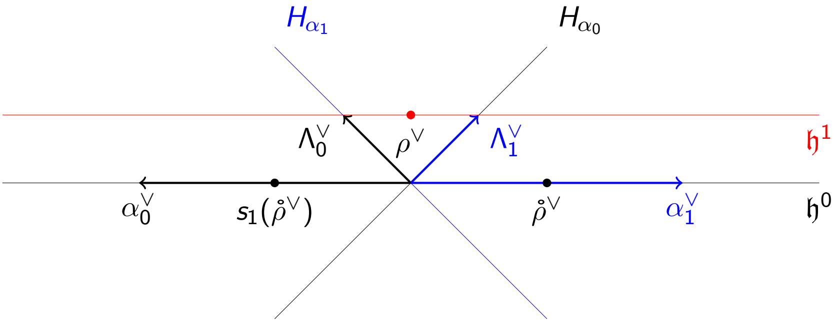
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on $\mathring{W}$*
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$, *not* $\mathrm{H}(\mathring{W})(0)$*!*

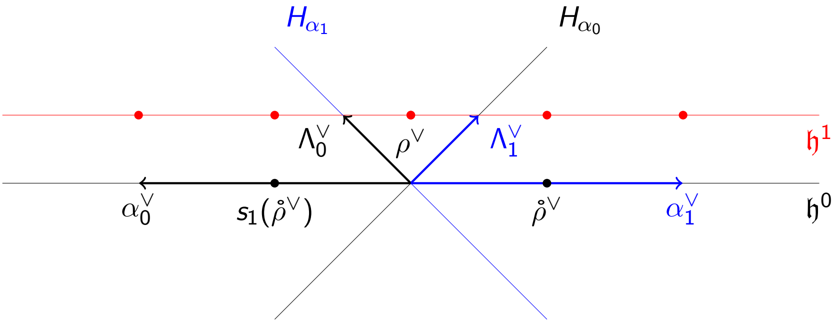
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on* $\mathring{W}$
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$, *not* $\mathrm{H}(\mathring{W})(0)$*!*

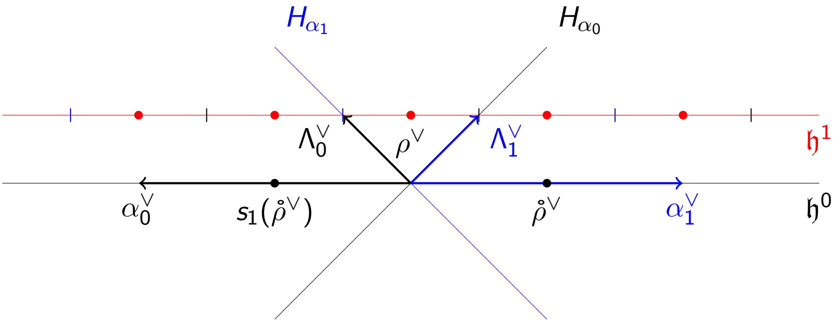
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on* $\mathring{W}$
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$*, not* $\mathrm{H}(\mathring{W})(0)$*!*

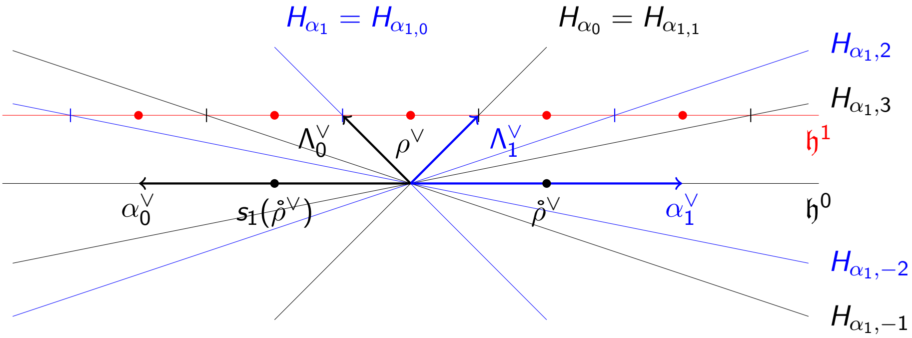
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on* $\mathring{W}$
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$*, not* $\mathrm{H}(\mathring{W})(0)$*!*

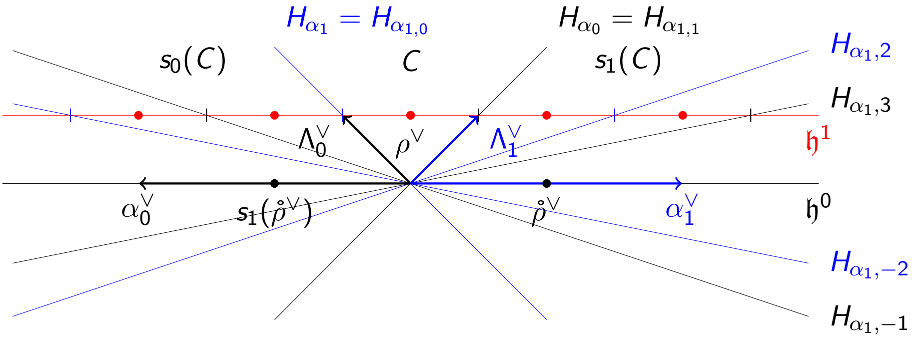
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on* $\mathring{W}$
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$, *not* $\mathrm{H}(\mathring{W})(0)$*!*

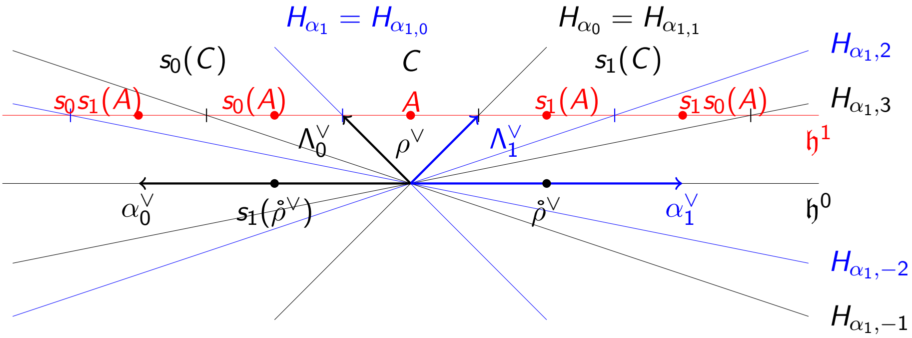
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on* $\mathring{W}$
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$, *not* $\mathrm{H}(\mathring{W})(0)$*!*

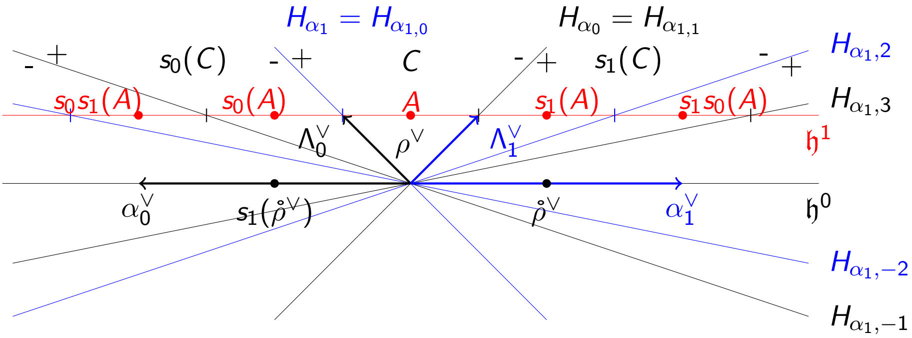
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- W *degenerates trivially to* $\mathring{W}$ *of type* A_1; $W = \mathring{W} \ltimes T$
- $\mathrm{H}(W)(0)$ *acts transitively on* $\mathring{W}$
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$, *not* $\mathrm{H}(\mathring{W})(0)$*!*

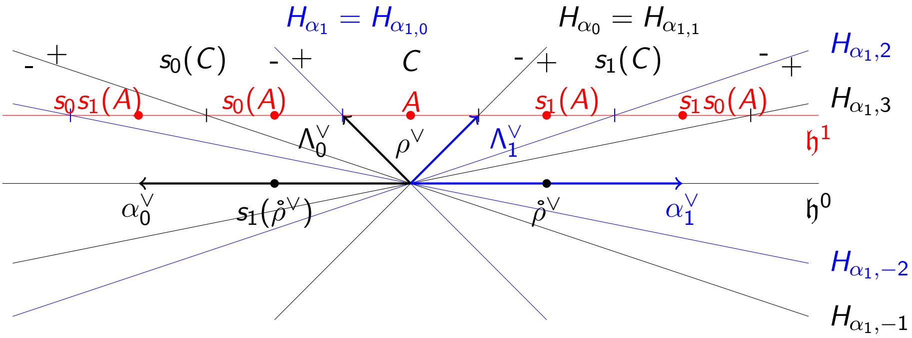
Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on* $\mathring{W}$
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$*, not* $\mathrm{H}(\mathring{W})(0)$*!*

Level 0 action of the type A_1^1 affine Hecke algebra



Remark (At level 0)

- *W degenerates trivially to $\mathring{W}$ of type A_1; $W = \mathring{W} \ltimes T$*
- $\mathrm{H}(W)(0)$ *acts transitively on* $\mathring{W}$
- $\mathrm{H}(W)(0)$ *degenerates to* $\mathrm{H}\mathring{W}$, *not* $\mathrm{H}(\mathring{W})(0)$*!*

Level 0 action of affine Hecke algebras

Theorem (H.,S.,T. 2008)

W: affine Weyl group
$\mathring{W}$: finite Weyl group induced by the level 0 action
At level 0:

- *W is quotiented to $\mathring{W}$*
- *$\pi_0, \pi_1, \pi_2, \ldots$ act transitively on $\mathring{W}$*
- *$\mathrm{H}(W)(0)$ is quotiented to $\mathrm{H}\mathring{W}$*
- *$\mathrm{H}(W)(q)$ is quotiented to $\mathrm{H}\mathring{W}$, for q generic*

Level 0 action of affine Hecke algebras

Theorem (H.,S.,T. 2008)

W: affine Weyl group
$\mathring{W}$: finite Weyl group induced by the level 0 action
At level 0:

- *W is quotiented to $\mathring{W}$*
- *$\pi_0, \pi_1, \pi_2, \ldots$ act transitively on $\mathring{W}$*
- *$\mathrm{H}(W)(0)$ is quotiented to $\mathrm{H}\mathring{W}$*
- *$\mathrm{H}(W)(q)$ is quotiented to $\mathrm{H}\mathring{W}$, for q generic*

Level 0 action of affine Hecke algebras

Theorem (H.,S.,T. 2008)

W: affine Weyl group
$\mathring{W}$: finite Weyl group induced by the level 0 action
At level 0:

- *W is quotiented to $\mathring{W}$*
- *$\pi_0, \pi_1, \pi_2, \ldots$ act transitively on $\mathring{W}$*
- *$\mathrm{H}(W)(0)$ is quotiented to $\mathrm{H}\mathring{W}$*
- *$\mathrm{H}(W)(q)$ is quotiented to $\mathrm{H}\mathring{W}$, for q generic*

Level 0 action of affine Hecke algebras

Theorem (H.,S.,T. 2008)

W: affine Weyl group
$\mathring{W}$: finite Weyl group induced by the level 0 action
At level 0:

- *W is quotiented to $\mathring{W}$*
- *$\pi_0, \pi_1, \pi_2, \ldots$ act transitively on $\mathring{W}$*
- *$\mathrm{H}(W)(0)$ is quotiented to $\mathrm{H}\mathring{W}$*
- *$\mathrm{H}(W)(q)$ is quotiented to $\mathrm{H}\mathring{W}$, for q generic*

Hecke group algebras and principal series representations

- $Y^{\lambda^\vee} \in \mathrm{H}(W)(q)$ (analog of translations in W)
- $\mathbb{C}[Y]$: commutative algebra, $\mathbb{C}[Y]^{\mathring{W}}$: center of $\mathrm{H}(W)(q)$
- t: character on $\mathbb{C}[Y]$
  - Central specialization of $\mathbb{C}[Y]^W$ on t:
    Quotient $\mathcal{H}(q,t)$ of $\mathrm{H}(W)(q)$ of dimension $|W|^2$
  - Representation $\rho_t := t \uparrow_{\mathbb{C}[Y]}^{\mathrm{H}(W)(q)}$ of $\mathrm{H}(W)(q)$ on $\mathbb{C}\mathring{W}$:
    Quotient of $\mathcal{H}(q,t)$, generically trivial

Theorem (S.,T., 2008)

Take q non zero and non root of unity, and $t : Y^{\lambda^\vee} \mapsto q^{-\operatorname{ht}(\lambda^\vee)}$
Then, $\rho_t(\mathrm{H}(W)(q)) = \mathrm{H}\mathring{W}$
I.e. $\mathrm{H}\mathring{W}$ (non trivial!) quotient of $\mathcal{H}(q,t)$ and of $\mathrm{H}(W)(q)$

- Proof: diagonalization of the action of Y on $\mathbb{C}\mathring{W}$, using alcove walks and the intertwining operators τ_i
- What happens at roots of unity?

Hecke group algebras and principal series representations

- $Y^{\lambda^\vee} \in \mathrm{H}(W)(q)$ (analog of translations in W)
- $\mathbb{C}[Y]$: commutative algebra, $\mathbb{C}[Y]^{\mathring{W}}$: center of $\mathrm{H}(W)(q)$
- t: character on $\mathbb{C}[Y]$
  - Central specialization of $\mathbb{C}[Y]^W$ on t: Quotient $\mathcal{H}(q,t)$ of $\mathrm{H}(W)(q)$ of dimension $|W|^2$
  - Representation $\rho_t := t \uparrow_{\mathbb{C}[Y]}^{\mathrm{H}(W)(q)}$ of $\mathrm{H}(W)(q)$ on $\mathbb{C}\mathring{W}$: Quotient of $\mathcal{H}(q,t)$, generically trivial

Theorem (S.,T., 2008)

Take q non zero and non root of unity, and $t : Y^{\lambda^\vee} \mapsto q^{-\operatorname{ht}(\lambda^\vee)}$
Then, $\rho_t(\mathrm{H}(W)(q)) = \mathrm{H}\mathring{W}$
I.e. $\mathrm{H}\mathring{W}$ (non trivial!) quotient of $\mathcal{H}(q,t)$ and of $\mathrm{H}(W)(q)$

- Proof: diagonalization of the action of Y on $\mathbb{C}\mathring{W}$, using alcove walks and the intertwining operators τ_i
- What happens at roots of unity?

Hecke group algebras and principal series representations

- $Y^{\lambda^\vee} \in \mathrm{H}(W)(q)$ (analog of translations in W)
- $\mathbb{C}[Y]$: commutative algebra, $\mathbb{C}[Y]^{\mathring{W}}$: center of $\mathrm{H}(W)(q)$
- t: character on $\mathbb{C}[Y]$
  - Central specialization of $\mathbb{C}[Y]^W$ on t:
    Quotient $\mathcal{H}(q,t)$ of $\mathrm{H}(W)(q)$ of dimension $|W|^2$
  - Representation $\rho_t := t \uparrow_{\mathbb{C}[Y]}^{\mathrm{H}(W)(q)}$ of $\mathrm{H}(W)(q)$ on $\mathbb{C}\mathring{W}$:
    Quotient of $\mathcal{H}(q,t)$, generically trivial

Theorem (S.,T., 2008)

Take q non zero and non root of unity, and $t : Y^{\lambda^\vee} \mapsto q^{-\operatorname{ht}(\lambda^\vee)}$
Then, $\rho_t(\mathrm{H}(W)(q)) = \mathrm{H}\mathring{W}$
I.e. $\mathrm{H}\mathring{W}$ (non trivial!) quotient of $\mathcal{H}(q,t)$ and of $\mathrm{H}(W)(q)$

- Proof: diagonalization of the action of Y on $\mathbb{C}\mathring{W}$, using alcove walks and the intertwining operators τ_i
- What happens at roots of unity?

Hecke group algebras and principal series representations

- $Y^{\lambda^\vee} \in \mathrm{H}(W)(q)$ (analog of translations in W)
- $\mathbb{C}[Y]$: commutative algebra, $\mathbb{C}[Y]^{\mathring{W}}$: center of $\mathrm{H}(W)(q)$
- t: character on $\mathbb{C}[Y]$
  - Central specialization of $\mathbb{C}[Y]^{W}$ on t:
    Quotient $\mathcal{H}(q,t)$ of $\mathrm{H}(W)(q)$ of dimension $|W|^2$
  - Representation $\rho_t := t \uparrow_{\mathbb{C}[Y]}^{\mathrm{H}(W)(q)}$ of $\mathrm{H}(W)(q)$ on $\mathbb{C}\mathring{W}$:
    Quotient of $\mathcal{H}(q,t)$, generically trivial

Theorem (S.,T., 2008)

Take q non zero and non root of unity, and $t : Y^{\lambda^\vee} \mapsto q^{-\operatorname{ht}(\lambda^\vee)}$
Then, $\rho_t(\mathrm{H}(W)(q)) = \mathrm{H}\mathring{W}$
I.e. $\mathrm{H}\mathring{W}$ (non trivial!) quotient of $\mathcal{H}(q,t)$ and of $\mathrm{H}(W)(q)$

- Proof: diagonalization of the action of Y on $\mathbb{C}\mathring{W}$, using alcove walks and the intertwining operators τ_i
- What happens at roots of unity?

Hecke group algebras and principal series representations

- $Y^{\lambda^\vee} \in \mathrm{H}(W)(q)$ (analog of translations in W)
- $\mathbb{C}[Y]$: commutative algebra, $\mathbb{C}[Y]^{\mathring{W}}$: center of $\mathrm{H}(W)(q)$
- t: character on $\mathbb{C}[Y]$
  - Central specialization of $\mathbb{C}[Y]^W$ on t:
    Quotient $\mathcal{H}(q,t)$ of $\mathrm{H}(W)(q)$ of dimension $|W|^2$
  - Representation $\rho_t := t \uparrow_{\mathbb{C}[Y]}^{\mathrm{H}(W)(q)}$ of $\mathrm{H}(W)(q)$ on $\mathbb{C}\mathring{W}$:
    Quotient of $\mathcal{H}(q,t)$, generically trivial

Theorem (S.,T., 2008)

Take q non zero and non root of unity, and $t : Y^{\lambda^\vee} \mapsto q^{-\mathrm{ht}(\lambda^\vee)}$
Then, $\rho_t(\mathrm{H}(W)(q)) = \mathrm{H}\mathring{W}$
I.e. $\mathrm{H}\mathring{W}$ (non trivial!) quotient of $\mathcal{H}(q,t)$ and of $\mathrm{H}(W)(q)$

- Proof: diagonalization of the action of Y on $\mathbb{C}\mathring{W}$, using alcove walks and the intertwining operators τ_i
- What happens at roots of unity?

Hecke group algebras and principal series representations

- $Y^{\lambda^\vee} \in \mathrm{H}(W)(q)$ (analog of translations in W)
- $\mathbb{C}[Y]$: commutative algebra, $\mathbb{C}[Y]^{\mathring{W}}$: center of $\mathrm{H}(W)(q)$
- t: character on $\mathbb{C}[Y]$
  - Central specialization of $\mathbb{C}[Y]^W$ on t:
    Quotient $\mathcal{H}(q,t)$ of $\mathrm{H}(W)(q)$ of dimension $|W|^2$
  - Representation $\rho_t := t \uparrow_{\mathbb{C}[Y]}^{\mathrm{H}(W)(q)}$ of $\mathrm{H}(W)(q)$ on $\mathbb{C}\mathring{W}$:
    Quotient of $\mathcal{H}(q,t)$, generically trivial

Theorem (S.,T., 2008)

Take q non zero and non root of unity, and $t : Y^{\lambda^\vee} \mapsto q^{-\operatorname{ht}(\lambda^\vee)}$
Then, $\rho_t(\mathrm{H}(W)(q)) = \mathrm{H}\mathring{W}$
I.e. $\mathrm{H}\mathring{W}$ (non trivial!) quotient of $\mathcal{H}(q,t)$ and of $\mathrm{H}(W)(q)$

- Proof: diagonalization of the action of Y on $\mathbb{C}\mathring{W}$, using alcove walks and the intertwining operators τ_i
- What happens at roots of unity?