Sorting monoids on Coxeter groups

A computer exploration with `Sage-Combinat`

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arXiv:0711.1561v1 [math.RT]
arXiv:0804.3781v1 [math.RT]
arXiv:0912.2212v1 [math.CO]
+ research in progress

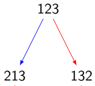
Sage-Combinat (`combinat.sagemath.org`)

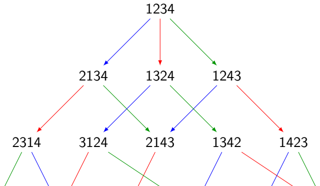
- 50+ research articles
- NSF Sponsored
- Sage: 300 tickets / 100k lines integrated in Sage
- MuPAD: 115k lines of MuPAD, 15k lines of C++, 32k lines of tests, 600 pages of doc
- Nicolas Borie, Daniel Bump, Jason Bandlow, Adrien Boussicault, Vincent Delecroix, Paul-Olivier Dehaye, Tom Denton, Dan Drake, Teresa Gomez Diaz, Mike Hansen, Ralf Hemmecke, Florent Hivert, Brant Jones, Sébastien Labbé, Yann Laigle-Chapuy, Andrew Mathas, Gregg Musiker, Steven Pon, Franco Saliola, Anne Schilling, Mark Shimozono, Nicolas M. Thiéry, Justin Walker, Qiang Wang, Mike Zabrocki, ...

Bubble (anti) sort algorithm

1234
1234
1243
1423
4123
4123
4132
4312
4312
4321
4321

Underlying combinatorics: right permutohedron





Coxeter groups

Definition (Coxeter group W)

Generators : $s_1, s_2, \ldots$ (simple reflections)

Relations: $s_i^2 = 1$ and $\underbrace{s_i s_j \cdots}_{m_{i,j}} = \underbrace{s_j s_i \cdots}_{m_{i,j}}$, for $i \neq j$

- Reduced word
- Length

Orders on words and on Coxeter group elements

Definition (Orders on words)

Let $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$:

- u **left factor** of v if $v = u_1 \cdots u_k \cdots$
- u **right factor** of v if $v = \cdots u_1 \cdots u_k$
- u **factor** of v if $v = \cdots u_1 \cdots u_k \cdots$
- u **subword** of v if $v = \cdots u_1 \cdots u_2 \cdots u_k \cdots$

Definition (Orders on Coxeter group elements)

- Right weak order
- Left weak order
- Left-right weak order
- Bruhat order

Blocks of permutations

Definition (Block of a permutation w)

- Type A: sub-permutation matrix
- Type free: J, K such that $W_J w = w W_K$

- Example: $w := 36475812$
- Simple permutation: cf. [Albert, Atkinson 05]
- $\{\texttt{blocks of } w\}$: sub-lattice of the Boolean lattice

Definition (HST09: Cutting poset $(W, \sqsubset)$)

$u \sqsubset w$ if $u = w^J$ with J block

- (almost) lattice
- Möbius function: inclusion-exclusion along minimal blocks

Hecke monoid

Definition (0-Hecke monoid $H_0(W)$ of a Coxeter group W)

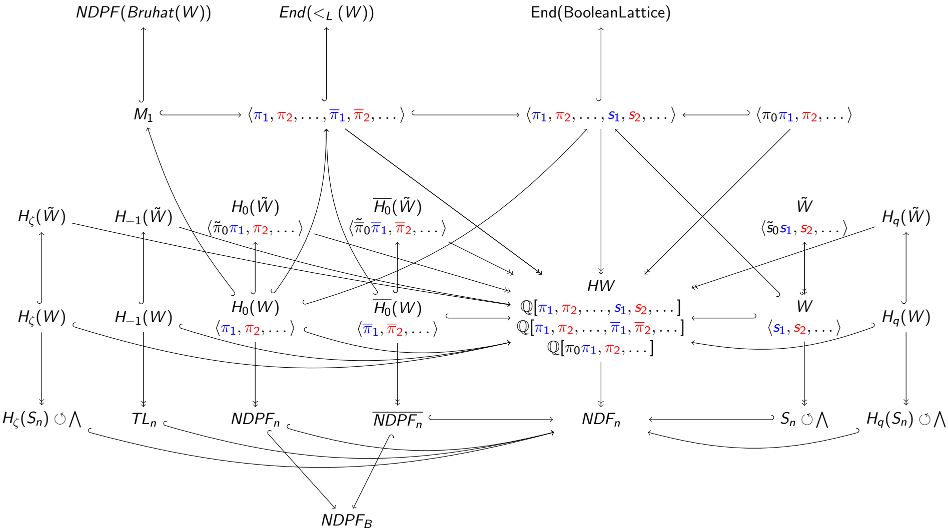
Generators : $\langle \pi_1, \pi_2, \ldots \rangle$ (simple reflections)
Relations: $\pi_i^2 = \pi_i$ and braid relations

Theorem

$|H_0(W)| = |W|$
$+$ *lots of nice properties*

Motivation: simple combinatorial model (bubble sort)
appears in Iwahori-Hecke algebras, Schur symmetric functions, Schubert, Kazhdan-Lusztig polynomials, and Macdonald, (affine) Stanley symmetric functions, mathematical physics, Schur-Weyl duality for quantum groups, representations of $GL(\mathbb{F}_q)$, ...

The Big Picture



The bi-Hecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots \rangle$
$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Theorem (HST08)

$M(W)$ *admits* $|W|$ *simple / indecomposable projective modules*

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w . \dim P_w$$

Representation theory of algebras

Module: vector space V with a morphism $M \mapsto \mathrm{End}(V)$
Simple module: V contains no nontrivial submodule
Indecomposable module: V cannot be written as $V = V_1 \oplus V_2$
Projective module: $V \oplus \cdots = \mathbb{C}[M] \oplus \cdots \oplus \mathbb{C}[M]$

Theorem (See e.g. Curtis-Reiner)

Simple modules $\leftrightarrow$ indecomposable projective modules
Dimension formula, ...

Key role of idempotents:

- eV projective module
- If $f = uev$ then fM is isomorphic to a submodule of eM

Representation theory of monoids

Definition (J-(pre)order)

$x \leq_J y$ iff $x = uyv$, for some $u, v \in M$
$x, y \in M$ are in the same J-class if $x \leq_J y$ and $y \leq_J x$
A J-class is regular iff it contains an idempotent

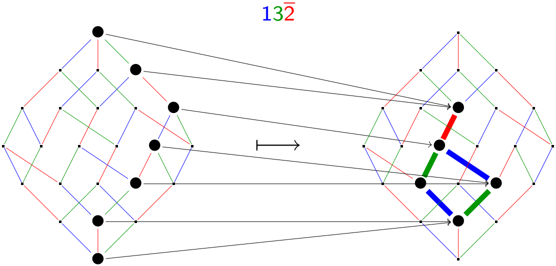
Theorem (See e.g. Ganyushkin, Mazorchuk, Steinberg 07)

The regular J-classes (essentially) determine the simple modules.

$\Longrightarrow$ Combinatorial Representation Theory

Key combinatorial lemma

Key combinatorial lemma



Lemma

For $f \in M(W)$ and $w \in W$: $(s_i w).f = w.f$ *or* $s_i\,(w.f)$

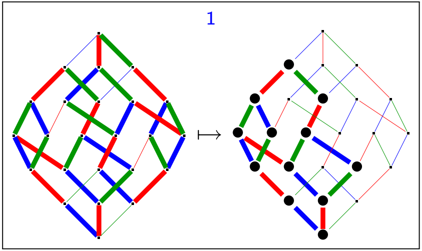
Proof.

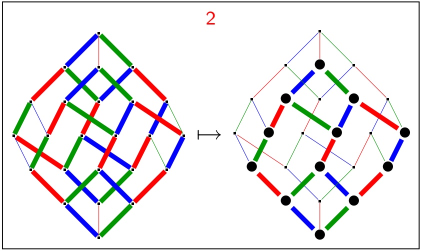
Exchange property / associativity □

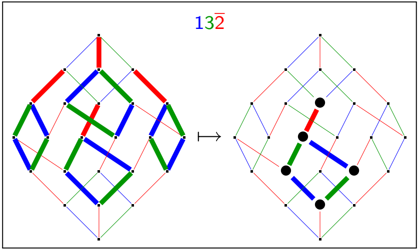
Corollary

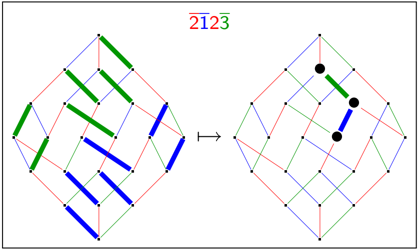
- *If $w = uv$, then $(uv).f = u'\,(v.f)$, where $u' <_B u$*
- *Preservation of left order: $u \leq v \implies u.f \leq v.f$*

Some elements of the monoid









Representation theory of $M(W)$

Theorem (HST'08)

$M(W)$ admits $|W|$ simple modules

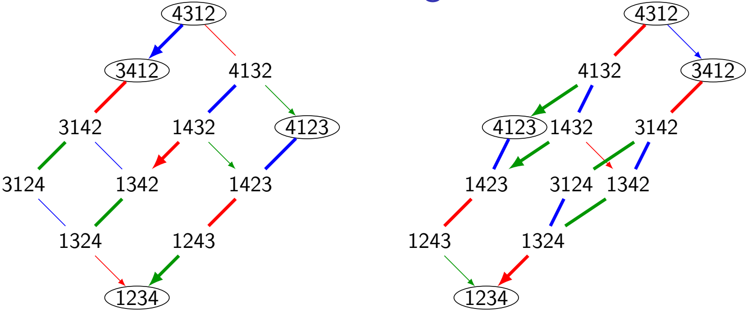
Sketch of proof.

- M acts transitively on intervals $[u, v]_L$
- The image set of an idempotent is an interval $[u, v]_L$
- $\exists!$ e_w idempotent with image set $[1, w]_L$, for any $w \in W$
- $(e_w)_{w \in W}$: transversal of the regular J-classes
  - $f = uev$ if and only if $\mathrm{im}(f)$ is a subinterval of $\mathrm{im}(e)$
- The groups $e_w M e_w$ are trivial □

Problem

Dimension of simple and projective modules?

Translation algebras



Definition (Translation algebra)

$T_w := \mathbb{Q}[\pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots]$ acting on $\mathbb{Q}.[1, w]_R$

- Blocks: $J = \{\}, \{1, 2\}, \{3\}, \{1, 2, 3\}$ $\Longrightarrow$ Submodules P_J
- T_w: max. algebra stabilizing all P_J $\Longrightarrow$ Repr. theory
- T_w quotient of $\mathbb{Q}[M(W)]$; top: simple module S_w of M
- Dimension: inclusion-exclusion along the cutting poset
- Generating series calculation?

The “Borel” submonoid M_1

Definition

Submonoid $M_1 := \{f \in M, f(1) = 1\}$

Properties (HST'09)

- *Generated by* e_w *for* w *grassmanian*
- *Weakly increasing and contracting on Bruhat* $\Longrightarrow$ *J-trivial*
- *Idempotents:* $(e_w)_{w \in W}$
- $|W|$ *simple modules of dimension* 1
- *Semi simple quotient: monoid algebra of* $(W, \vee_L)$
- *Conjugacy order among idempotents:* $<_L$
- $\dim P_w = |\{f \in M_1, f(w) = w...\}|$?

Problem

Inducing these results to M*?*

Representation theory of J-trivial monoids

Theorem (HST'09)

Combinatorial description of:

- *Simple modules*
- *Projective modules*
- *Cartan matrix*
- *Quiver*
- *q-Cartan matrix (in progress)*

in term of some statistic on M

Question

Induction from "Borel" submonoids?

The path algebra of a Quiver

Definition

- Quiver: (edge labeled) graph $Q = (V, E)$
- path of length l (possibly $= 0$)

$$p := (v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_l} v_l)$$

  such that e_i is an edge from v_{i-1} to v_i.
- path algebra (category): product = concatenation if last and first vertex matches else 0.

TODO : Add an example

Structure theorem for finite dimensional algebras

Definition

admissible ideal: included in the ideal of path of length ≥ 2.

Theorem

For any (elementary) algebra A, there is a **unique quiver** *Q such that A is the quotient of $\mathbb{C}Q$ by an admissible ideal I.*

Elementary algebras: simple module are all 1-dimensional.

Note: first order approximation of the algebra.

Note: the ideal I is far from being unique.

Vertices of the Quiver ?

Decomposition of the identity:

$$1 = \sum_e f_e \qquad \text{and} \qquad f_e f_{e'} = \delta_{ee'} f_e;$$

Theorem (HST 09)

Association: $e \in M \mapsto f_e \in \mathbb{C}[M]$, *such that*
Moreover $f_e = e +$ *smaller terms...*

Theorem

The vertices of the Quiver are naturally indexed by the idempotents of the monoid.

Cartan's invariants

Matrix decomposition of the algebra $x \in \mathbb{C}[M]$:

$$x = \sum_{e_1, e_2} x_{e_1,e_2} \quad \text{where} \quad x_{e_1,e_2} = f_{e_1} x f_{e_2} \,.$$

$$(xy)_{e_1,e_2} = \sum_{e} x_{e_1,e} y_{e,e_2}$$

Automorphism sub-monoids and factorizations

Definition (Automorphism sub-monoids)

$$\mathrm{rAut}(x) := \{u \in M \mid xu = x\}$$

Proposition

There exists a unique idempotent $\mathrm{rfix}(x)$ *such that*

$$\mathrm{rAut}(x) = \{u \in M \mid \mathrm{rfix}(x) \leq_J u\}\,.$$

Same one the left $(\mathrm{lAut}(x), \mathrm{lfix}(x))$.

Corollary: Cartan invariants + vertex of the quiver

Theorem

Cartan's invariants:

$$\dim(f_{e_1}\mathbb{C}[M]f_{e_2}) = \#\{x \in M \mid \operatorname{lfix}(x) = e_1 \textit{ and } \operatorname{rfix}(x) = e_2\}\,.$$

Factorizations

Definition

Let $x \in M$ non idempotent and $e :=$ and $f := \mathrm{rfix}(x)$.
A factorization $x = uv$ is

- **non-trivial** if $eu \neq e$ and $vf \neq f$
  equivalently if $u \notin \mathrm{lAut}(x)$ and $v \notin \mathrm{rAut}(x)$;
- **compatible** if u and v are non-idempotent and

$$\mathrm{lfix}(u) = e\,, \quad \mathrm{rfix}(v) = f \quad \text{and} \quad \mathrm{rfix}(u) = \mathrm{lfix}(v)\,;$$

Factorizations and irreducible

Proposition

- *compatible* $\Rightarrow$ *non-trivial;*
- *non-trivial and j-minimal* $\Rightarrow$ *compatible.*

Definition

$x \in M$ non idempotent is **irreducible** if there is no non-trivial factorizations $x = uv$.

The Quiver of (the algebra of) a j-trivial monoid

Theorem

The quiver of the algebra of M is the following:

- *There is one vertex v_e for each idempotent e of the monoid;*
- *For each irreducible element x in the monoid there is an arrow from $v_{\mathrm{lfix}(x)}$ to $v_{\mathrm{rfix}(x)}$.*

Sage : generic Algo + examples...

Conclusion

General strategy:

- Find combinatorial models for algebras and representations
- As simple as possible, but no simpler
- Concrete and effective
- Use representation theory and computer exploration as a guide
- Find the right point of view where proofs become trivial

Discussions?

- Integration of Jean-Éric's Semigroupe package into Sage
- Simple permutations and cutting poset
- Endomorphisms of the Boolean lattice