Théorie des représentations effective des monoïdes apériodiques

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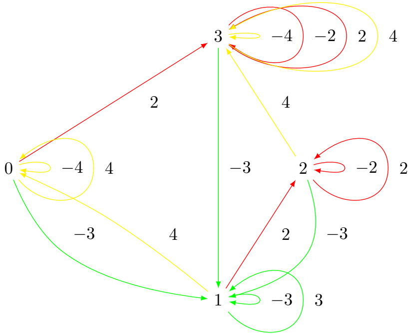

Combinatorial Representation Theory (1)

Representation theory: lots of natural numbers !

- dimension of simple and indecomposable projective modules ($\mathfrak{S}_n, \mathrm{gl}$: Kostka numbers)
- induction and restrictions multiplicities ($\mathfrak{S}_m \times \mathfrak{S}_n \to \mathfrak{S}_{m+n}$: Littlewood-Richardson rules)
- Cartan invariant matrices and quivers ($H_n(0)$: counting permutation by descents and recoils)
- decomposition map ($H_n(q \mapsto 0)$: counting tableaux by shape and descents)

Combinatorial Representation Theory (2)

Mostly effective: computer exploration !

Depending on

- the base field ($\mathbb{Q}$ or some extension)
- the sparsity of the multiplication table
- ...

Dimension up to 50 to 2000

Several recent examples are monoid algebras

- 0-Hecke algebras (Norton, Carter, Krob-Thibon, Duchamp-Hivert-Thibon, Fayers, Denton)
- Non-decreasing parking function (Denton-Hivert-Schilling-T)
- Solomon-Tits algebras (Schocker, Saliola)
- Left Regular Bands (Brown) ...

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. . . but this fact is seldom used . . .

Goals of the talk

- Describe the combinatorics of aperiodic monoids

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- Link with the representation theory
- Derive algorithm for computing the Cartan matrix

A simple example: Order preserving functions on the chain

Definition

$f : \{1, \dots, n\} \longmapsto \{1, \dots, n\}$ is **order preserving** if:

$$i \leq j \implies f(i) \leq f(j)$$

Example

The order preserving functions on $\{1 < 2 < 3\}$:

$$\{111, 112, 113, 122, 123, 133, 222, 223, 233, 333\}$$

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Remark

If f, g are order preserving, then so is fg.
Hence, the set $\mathcal{O}_n$ of such functions is a **monoid** *!*

This still works if $\leq$ is replaced by a partial order

Understanding the multiplication?

First approach: the multiplication table:

$*$	111	112	113	122	123	133	222	223	233	333
111	111	111	111	111	111	111	222	222	222	333
112	111	111	111	112	112	113	222	222	223	333
113	111	112	113	112	113	113	222	223	223	333
122	111	111	111	122	122	133	222	222	233	333
123	111	112	113	122	123	133	222	223	233	333
133	111	122	133	122	133	133	222	233	233	333
222	111	111	111	222	222	333	222	222	333	333
223	111	112	113	222	223	333	222	223	333	333
233	111	122	133	222	233	333	222	233	333	333
333	111	222	333	222	333	333	222	333	333	333

The Cayley graph of a monoid

Remark

Thanks to associativity, it is sufficient to consider products

$$xg, \quad \textit{for } x \in M \textit{ and } g \textit{ a generator}$$

Definition (Cayley graph)

Graph with edges $x \stackrel{g}{\longmapsto} xg$

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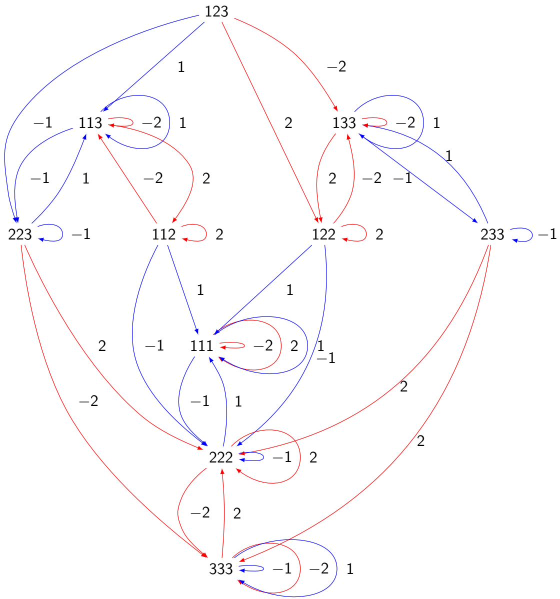
Example

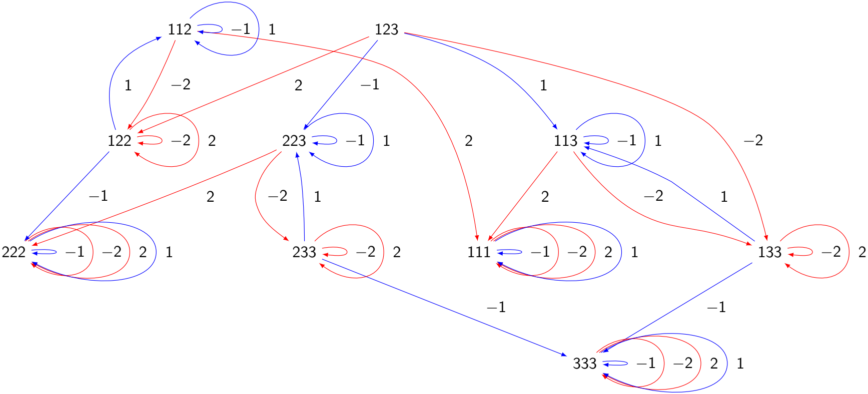
Canonical generators for $\mathcal{O}_3$:

$$\pi_1 = 113, \qquad \pi_2 = 122$$
$$\pi_{-1} = 133, \qquad \pi_{-2} = 223$$

The right Cayley graph of $\mathcal{O}_3$



The left Cayley graph of $\mathcal{O}_3$



Combinatorial Module X of M

Definition (Combinatorial Module)

Finite set X with an action of M on X

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Finite set X with an action of M on X

Described by its Cayley graph (an automaton)
Equivalently: representation of M as monoid of functions in X^X

Example

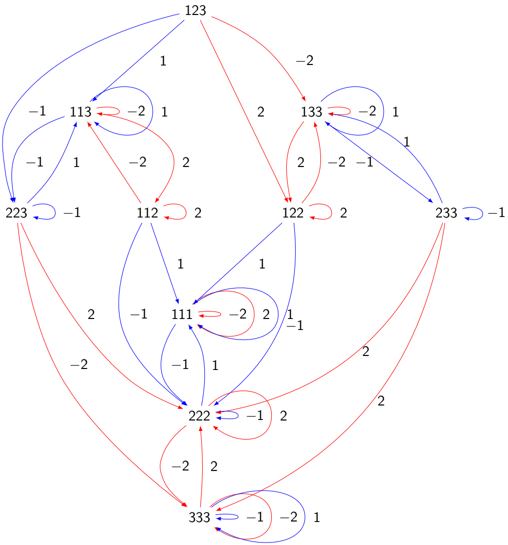
Regular representation of M acting on $X = M$ (associativity!)

Problem

Describe all modules / representations of M?

Submodules

$X' \subset X$ is a **submodule** if it is stable under the action of M



R-preorder (Green 50)

Definition ($\mathcal{R}$-preorder)

$$x \leq_R y \quad \text{if} \quad x \in yM$$

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- $\mathcal{R}$-**class** $\mathcal{R}(x)$: strongly connected component
- $\mathcal{R}$-**order** on $\mathcal{R}$-classes
- $\mathcal{R}$-**trivial monoid**: all $\mathcal{R}$-classes are trivial

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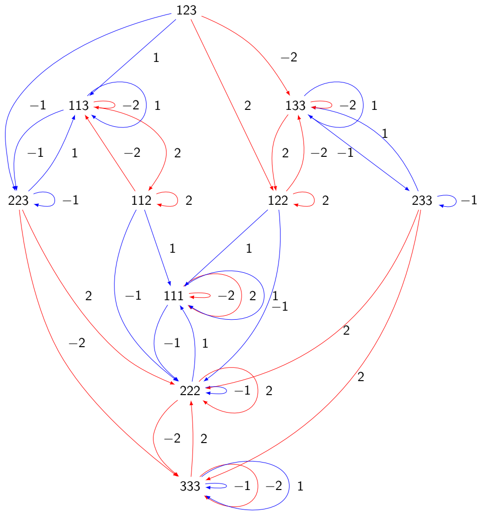
Submodule of X:

- Union of $\mathcal{R}$-classes of X
- Order ideal in $\mathcal{R}$-preorder

Quotients by submodules

Quotient by submodule $X' \subset X$: $X \backslash X' \cup \{\emptyset\}$



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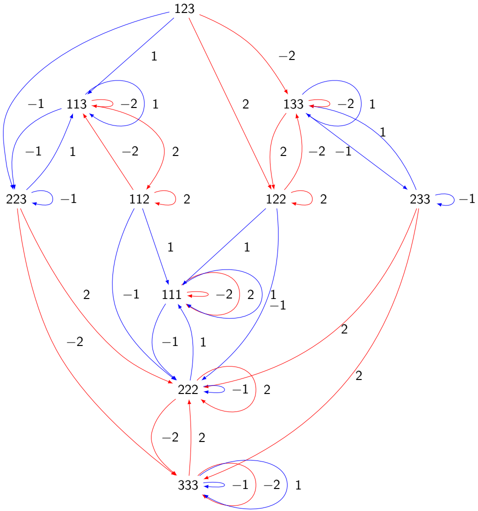

$\mathcal{R}$-classes: smallest subquotients

Left-right Cayley graph, $\mathcal{J}$-preorder

Problem

- *Why do we get several times the same module?*
- *Can we exploit associativity?*

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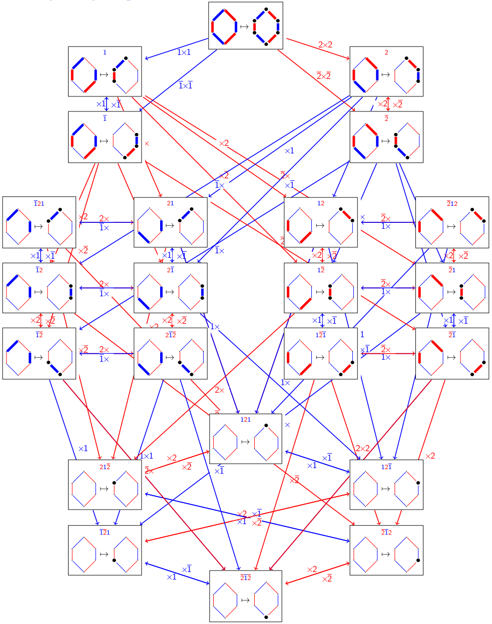
- **Left-right Cayley graph**
- $\mathcal{J}$**-class**
- $\mathcal{J}$**-preorder**
- $\mathcal{J}$**-trivial**

Example: the left-right Cayley graph for $\mathcal{O}3$

The left-right Cayley graph for the biHecke monoid for $\mathfrak{S}_2$



The eggbox picture

Proposition

Let J be a $\mathcal{J}$-class. Then,

$$J \quad \approx_{M-\mathrm{mod}-M} \quad L \times R$$

where L and R are respectively left and right classes

If e is an idempotent:

$$\mathcal{J}(e) = \mathcal{L}(e)\mathcal{R}(e)$$

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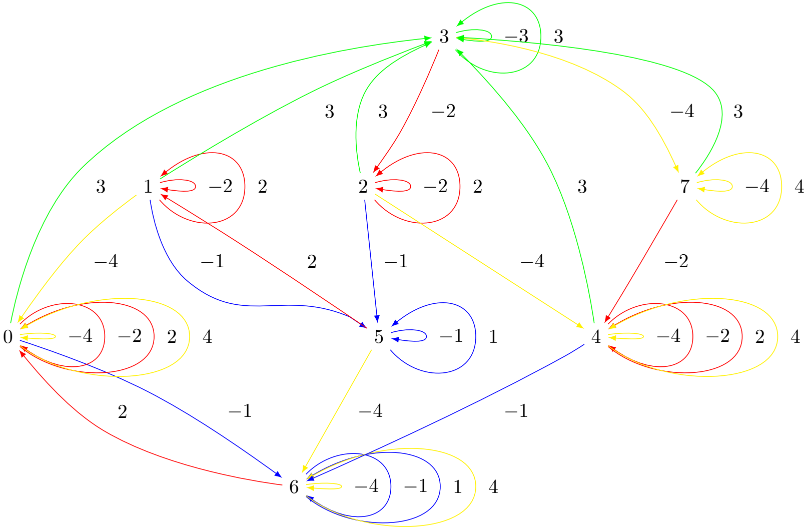
Note: unless M is **aperiodic**: there are in fact groups in the boxes

Example: a left module for the biHecke monoid for $\mathfrak{S}_5$



Linear representations

Definition (Module)

Vector space V with an action of M on V by linear operators

Equivalently: linear representation of M as submonoid of $M_n(K)$

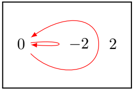
Example

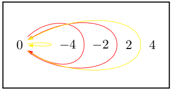
Linear module $V = \mathbb{Q}X$ associated to X

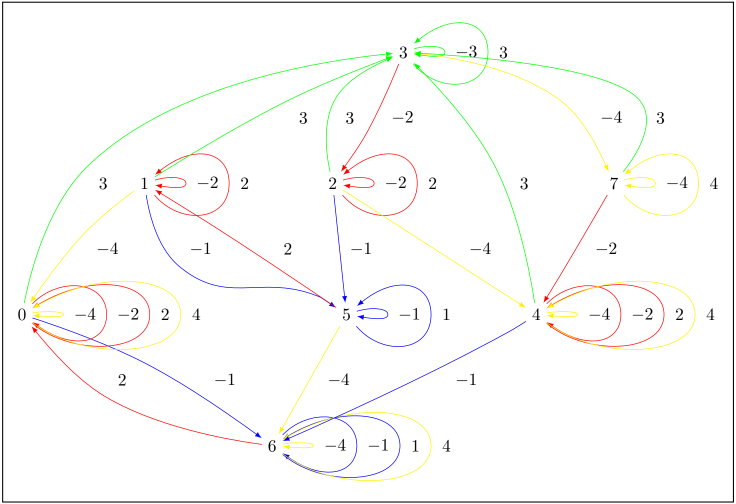
Definition

- Submodule, simple module
- Quotient module

Finding submodules?





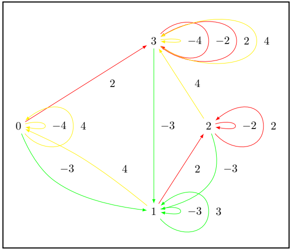


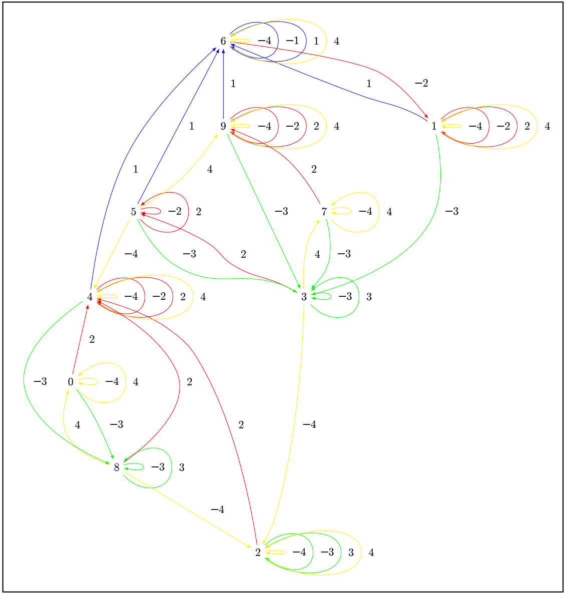
Projective module associated to a simple module S

Definition

S together with everything that can appear below S when embedding S into another module M

Embedding submodules by linear algebra





What's known about linear representations?

Finite groups

- Semi-simple: simple = projective
- Character theory
- Fast $o(n)$ algorithms

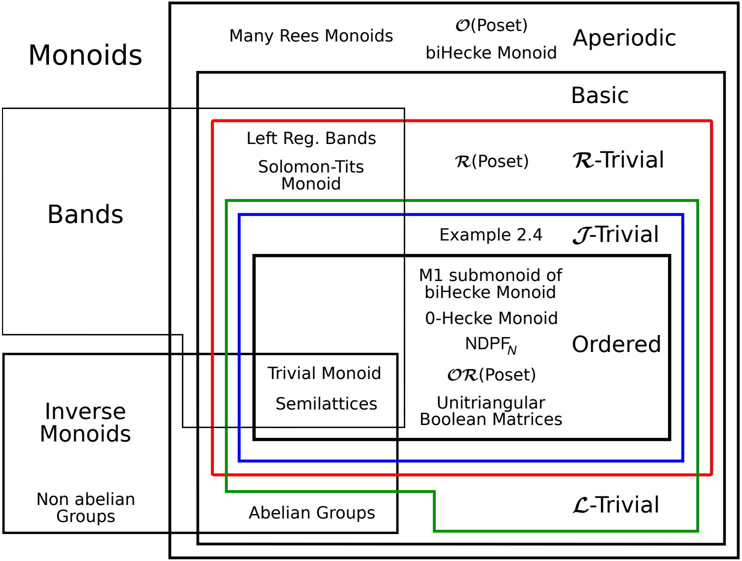
Finite dimensional algebras

- One-to-one correspondance Simple - Projective modules
- Algorithmic: minimal polynomial, linear algebra: $O(n^3)$
- In practice: dimension ≤ 1000

Monoids

- In progress, thanks to Franco :-)

Zoology of monoids



Goal for the rest of the talk

For an aperiodic monoid, calculate

- **Cartan matrix**
- Projective modules
- Quiver
- Radical / socle filtration

Analogue of the $\mathcal{R}$-preorder

Definition (Composition series)

$$\{0\} = V_0 \subset \cdots \subset V_\ell = V$$

such that V_{k+1}/V_k is simple

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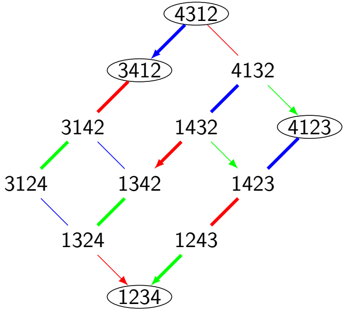

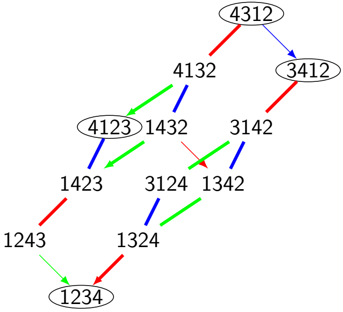

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Proposition

Composition series are not unique
The multiset $\{\{[V_{k+1}/V_k]\}\}$ *of the factors is*

Analogue of $\mathcal{J}$-preorder

Definition

A: finite dimensional algebra (e.g. $A = \mathbb{Q}[M]$)
A is an A-mod-A module (or $A^{\mathrm{op}} \otimes A$-module)
Composition series: $\{0\} = A_0 \subset \cdots \subset A_\ell = A$

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Proposition (Analogue of the eggbox picture)

$$A_{k+1}/A_k \approx_{A-\mathrm{mod}-A} L \otimes R$$

where L is a simple left module and R is a simple right module

Cartan matrix

Definition

$C = (c_{i,j})_{i,j}$, with:

$$c_{i,j} \quad = \quad |\{k,\ A_{k+1}/A_k \ \approx_{A-\mod -A}\ S_i \otimes S_j\}|$$

Equivalent definitions:

- On the left: $[P_j] = \sum_i c_{i,j}[S_i]$
- On the right: $[P_i] = \sum_j c_{i,j}[S_j]$
- Dimension of sandwich by idempotents: $c_{i,j} = \dim e_i A e_j$

Usual approach by orthogonal idempotents

1. Build a decomposition of the identity into orthogonal idempotents e_i
2. Compute $e_i A e_j$
3. Build the projective modules as $e_i A$

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Problem

Non trivial construction!

- 0-*Hecke in type A: combinatorial formula [Denton'10]*
- $\mathcal{R}$*-trivial: recursive formula [Berg,Bergeron,Bhargava,Saliola'10]*
- *Aperiodic?*
- *Algebra: may require arbitrary algebraic extensions*

Idempotent free approach?

Special case: $\mathcal{J}$-trivial monoids [Denton,Hivert,Schilling,T'11]

Each $\mathcal{J}$-class $\{x\}$ gives a simple $A-\mathrm{mod}-A$ module

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$$c_{i,j} = |\{x, \quad \text{lfix}(x) = i, \text{rfix}(x) = j\}|$$

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Idem for the radical and quiver.

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Problem

Radical filtration?

Cartan matrix of aperiodic monoids using the eggbox picture

Remark

- *The composition series of $\mathbb{Q}[M]$ refines the decomposition of M into $\mathcal{J}$-classes*
- *For J a $\mathcal{J}$-class of the form $L \times R$:*

$$J \quad \approx_{\mathbb{Q}[M]-\text{mod}-\mathbb{Q}[M]} \quad \mathbb{Q}L \otimes \mathbb{Q}R$$

Proposition (T.)

M_L: decomposition matrix of left modules into simples
M_R: decomposition matrix of right modules into simples
Then, $C = M_L M_R$

Remark: M_L is upper triangular; M_R is lower triangular

Algorithm

1. Compute representatives of left and right class modules

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   - The eggbox matrix encodes the product (Rees-matrix monoid)
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3. Compute the composition series of each type of left and right class module:
   Find all embeddings of the simple modules; quotient out; repeat
4. Gather all the results and calculate $C = M_L M_R$

Consequences

- No algebraic extension of $\mathbb{Q}$ needed (expected)
- Split the linear algebra in small chunks (parallelization, ...)
- Take advantage of the redundance
- Computation of the representation theory of a monoid of size 47000 in two hours

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But: not purely combinatorial

- Description?
- Quiver?
- Socle/Radical filtration?
- Interesting examples in combinatorics?
- Generalization to any finite monoid?