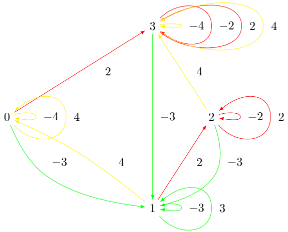
Representation theory of finite aperiodic monoids and of the biHecke monoid of a Coxeter group

T. Denton[2] F. Hivert[1] A. Schilling[2] Nicolas M. Thiéry[3]

[1]LITIS, Université Rouen, France

[2]University of California at Davis, USA

[3]Laboratoire de Mathématiques d'Orsay, Université Paris Sud, France



Abstract

Unlike for groups and algebras, the systematic study of the representation theory of finite monoids is a fairly recent area of research, and is witnessing rapid progress. This is of great interest for algebraic combinatorics, as many algebras arising there are monoid algebras; studying their representation theory, especially with the help of computer exploration, is a powerful tool to unravel more combinatorics out of them. In this talk, we present some results and algorithms about the representation theory of J-trivial and aperiodic monoids. We use as running example our favorite toy that spawned our interest in the topic: the biHecke monoid of a Coxeter group W. Along the way we unravel a new distributive semi-lattice on W, which gives a generalization of the combinatorics of descents to any interval in right order.

On the representation theory of finite aperiodic monoids ... work in progress ...
On the representation theory of finite J-trivial monoids Joint with T. Denton, F. Hivert, and A. Schilling arXiv:1010.3455v3 [math.RT]

The biHecke monoid of a finite Coxeter group and its representations Joint with F.

Hivert, and A. Schilling arXiv:1012.1361v1 [math.CO]

Combinatorial Representation Theory I

Representation theory: lots of natural numbers !

- dimension of simple and indecomposable projective modules ($\mathfrak{S}_n, \mathrm{gl}_n$: Kostka numbers)
- induction and restrictions multiplicities ($\mathfrak{S}_m \times \mathfrak{S}_n \to \mathfrak{S}_{m+n}$: Littlewood-Richardson rules)
- Cartan invariant matrices and quivers ($H_n(0)$: counting permutation by descents and recoils)
- decomposition map ($H_n(q \mapsto 0)$: counting tableaux by shape and descents)

Combinatorial Representation Theory II

Mostly effective: computer exploration !

Depending on

- the base field ($\mathbb{Q}$ or some extension)
- the sparsity of the multiplication table
- ...

Dimension up to 50 to 2000

Several recent examples are monoid algebras

- 0-Hecke algebras (Norton, Carter, Krob-Thibon, Duchamp-Hivert-Thibon, Fayers, Denton)
- Non-decreasing parking function (Denton-Hivert-Schilling-T)
- Solomon-Tits algebras (Schocker, Saliola)
- Left Regular Bands (Brown) ...

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Goals of the talk

- Describe the combinatorics of aperiodic monoids
- Link with the representation theory
- Derive an algorithm for computing the Cartan matrix
  And some more!

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Running example: Order preserving functions on the chain

Definition

$f : \{1, \ldots, n\} \longmapsto \{1, \ldots, n\}$ is **order preserving** if:

$$i \leq j \implies f(i) \leq f(j)$$

Example

The order preserving functions on $\{1 < 2 < 3\}$:

$$\{111, 112, 113, 122, 123, 133, 222, 223, 233, 333\}$$

Remark

If f, g are order preserving, then so is fg.
Hence, the set $\mathcal{O}_n$ of such functions is a **monoid** *!*

This still works if $\leq$ is replaced by a partial order

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Understanding the multiplication?

First approach: the multiplication table:

*	111	112	113	122	123	133	222	223	233	333
111	111	111	111	111	111	111	222	222	222	333
112	111	111	111	112	112	113	222	222	223	333
113	111	112	113	112	113	113	222	223	223	333
122	111	111	111	122	122	133	222	222	233	333
123	111	112	113	122	123	133	222	223	233	333
133	111	122	133	122	133	133	222	233	233	333
222	111	111	111	222	222	333	222	222	333	333
223	111	112	113	222	223	333	222	223	333	333
233	111	122	133	222	233	333	222	233	333	333
333	111	222	333	222	333	333	222	333	333	333

The Cayley graph of a monoid

Remark

Thanks to associativity, it is sufficient to consider products

$$xg, \quad \textit{for } x \in M \textit{ and } g \textit{ a generator}$$

Definition (Cayley graph)

Graph with edges $x \stackrel{g}{\longmapsto} xg$

Example

Canonical generators for $\mathcal{O}_3$:

$$\pi_1^+ = 223, \qquad \pi_2^+ = 133$$
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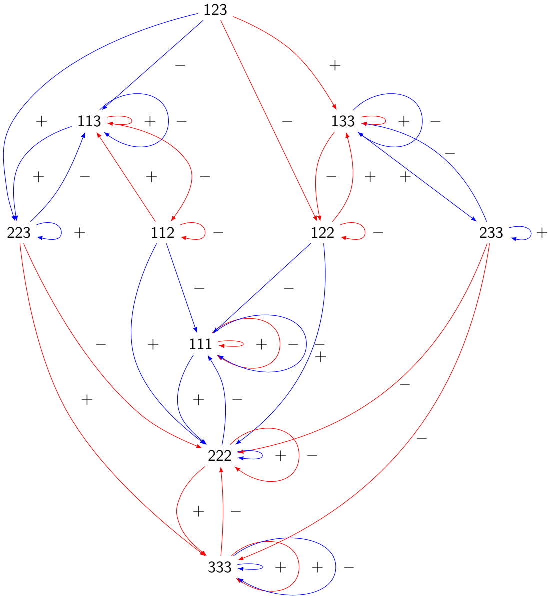
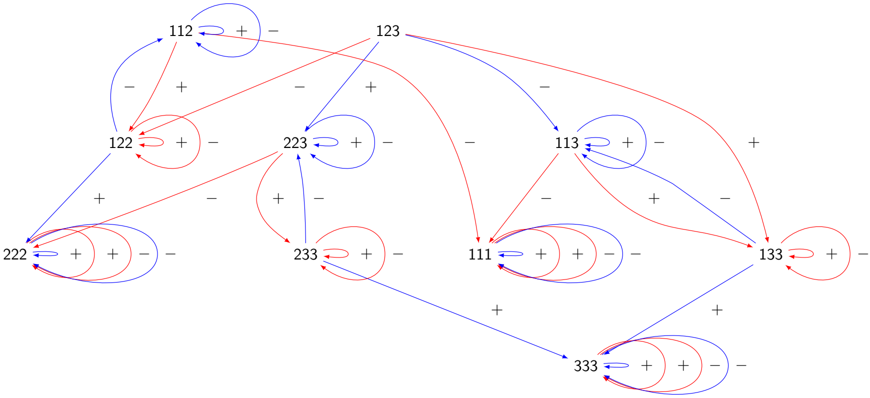
The right Cayley graph of $\mathcal{O}_3$



The left Cayley graph of $\mathcal{O}_3$



Combinatorial Module X of M

Definition (Combinatorial Module)

Finite set X with an action of M on X

Described by its Cayley graph (an automaton)
Equivalently: representation of M as monoid of functions in X^X

Example

Regular representation of M acting on $X = M$ (associativity!)

Problem

Describe all modules / representations of M?

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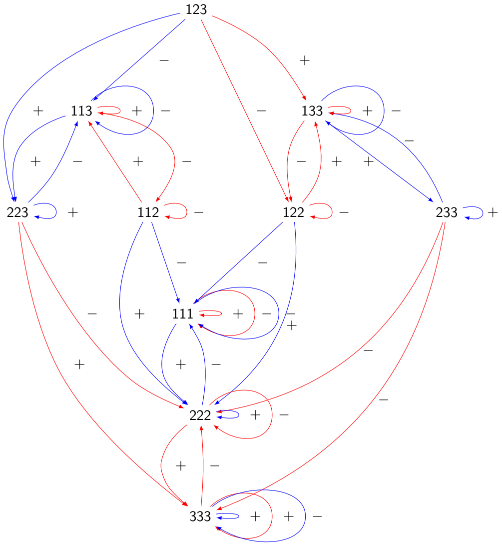
Submodules

$X' \subset X$ is a **submodule** if it is stable under the action of M



R-preorder (Green 50)

Definition ($\mathcal{R}$-preorder)

$$x \leq_R y \quad \text{if} \quad x \in yM$$

- $\mathcal{R}$-**class** $\mathcal{R}(x)$: strongly connected component
- $\mathcal{R}$-**order** on $\mathcal{R}$-classes
- $\mathcal{R}$-**trivial monoid**: all $\mathcal{R}$-classes are trivial

Submodule of X:

- Union of $\mathcal{R}$-classes of X
- Order ideal in $\mathcal{R}$-preorder

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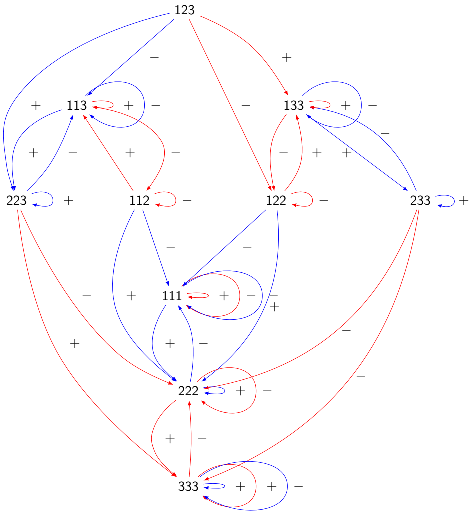

$\mathcal{R}$-classes: smallest (combinatorial) subquotients

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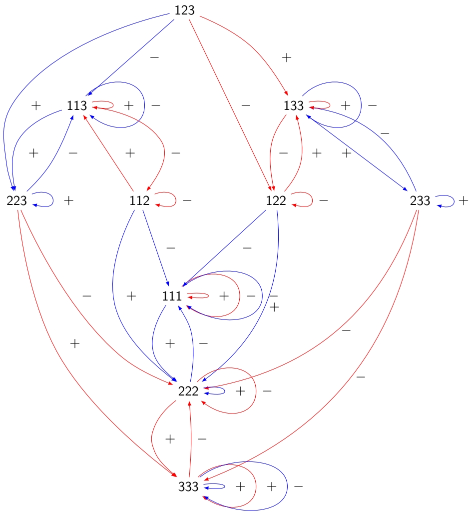

$\mathcal{R}$-classes: smallest (combinatorial) subquotients

Left-right Cayley graph, $\mathcal{J}$-preorder

Problem

- *Why do we get several times the same module?*
- *Can we exploit associativity?*

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Example: the left-right Cayley graph for $\mathcal{O}3$

The eggbox picture

Proposition

Let J be a $\mathcal{J}$-class. Then,

$$J \quad \approx_{M-\text{mod}-M} \quad L \times R$$

where L and R are respectively left and right classes

If e is an idempotent:

$$\mathcal{J}(e) = \mathcal{L}(e)\mathcal{R}(e)$$

Note: unless M is **aperiodic**: there are in fact groups in the boxes

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Motivation: Schubert calculus, symmetric function

Divided differences operators:

$$\partial_i f(x_1, \dots, x_n) := \frac{f(\dots, x_i, x_{i+1}, \dots) - f(\dots, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}}$$

$$\pi_i := \partial_i x_i, \qquad \hat{\pi}_i := 1 - \pi_i, \qquad s_i \dots$$

Problem

All these families satisfy the braids relations
Describe the mixed relations?

Applications

- (Algorithmic) exploitation of symmetries
- Schur, Schubert, Macdonald, Kazhdan-Lusztig polynomials, (affine) Stanley symmetric functions
- Mathematical physics, Representation theory, Probability, …

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Bubble (anti) sort algorithm

1234

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1243

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1423

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4123

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Underlying combinatorics: right permutahedron

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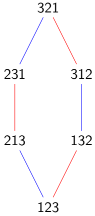
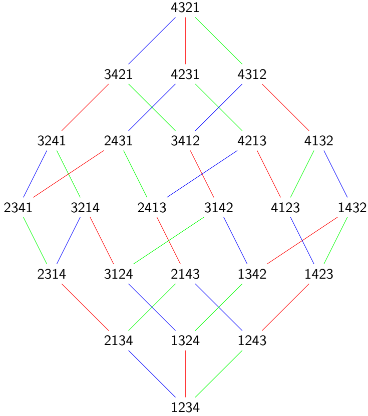
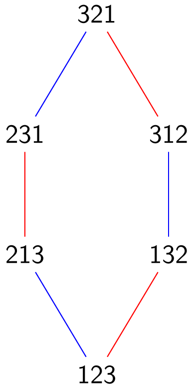
Underlying combinatorics: right permutahedron





The permutohedron, as an automaton



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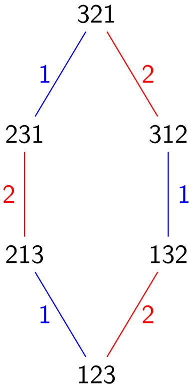

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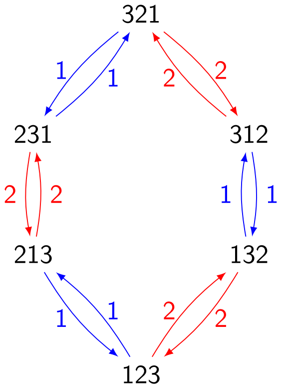

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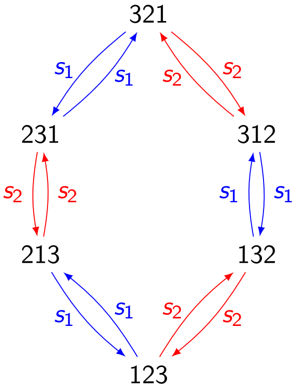

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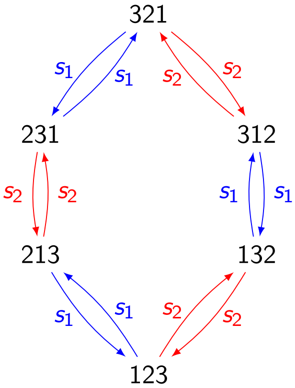

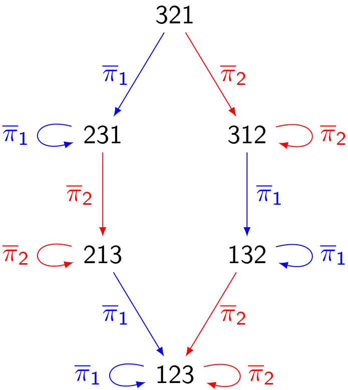

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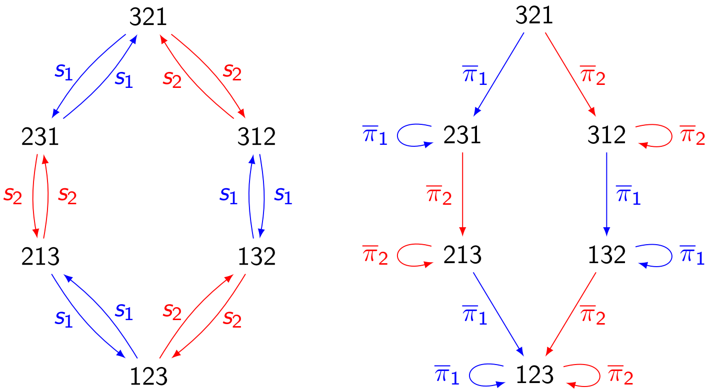

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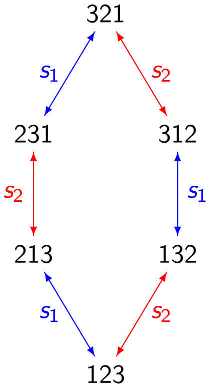

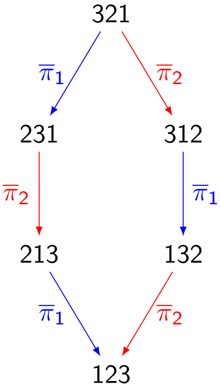

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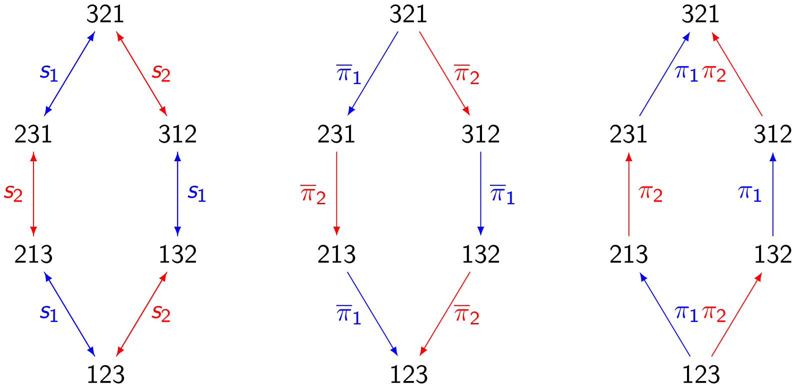
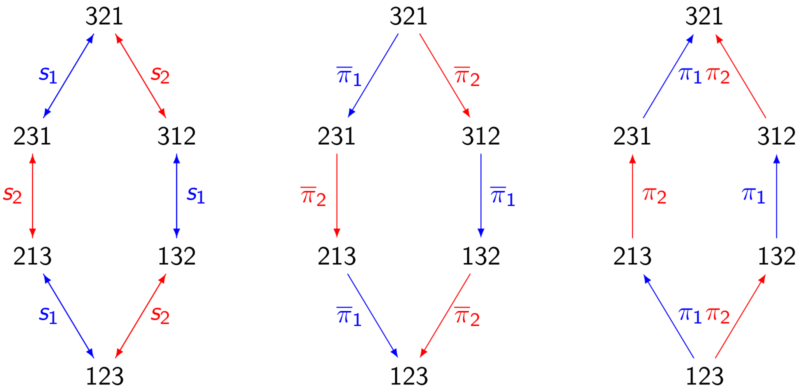


Monoids associated to the permutohedron



$$s_i^2 = 1$$
$$s_1 s_2 s_1 = s_2 s_1 s_2$$

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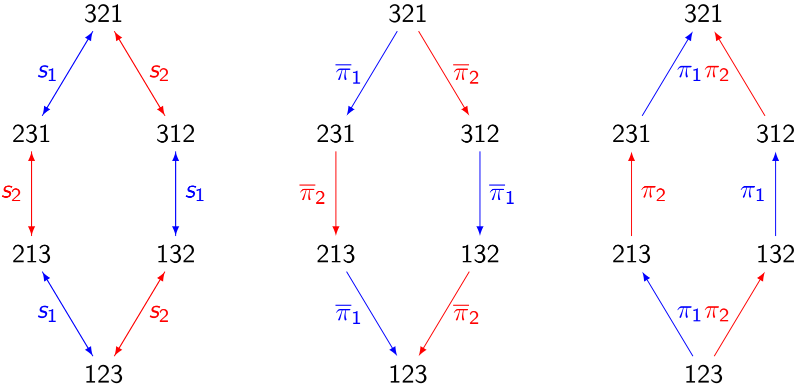

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Symmetric group $\mathfrak{S}_3$

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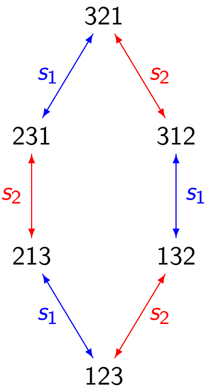

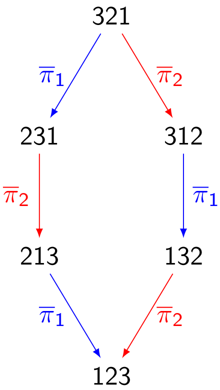

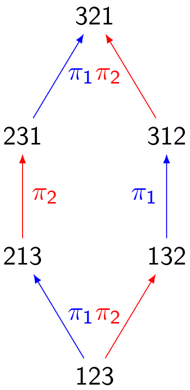

$s_i^2 = 1$

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$\overline{\pi}_i^2 = \overline{\pi}_i$

$\overline{\pi}_1\overline{\pi}_2\overline{\pi}_1 = \overline{\pi}_2\overline{\pi}_1\overline{\pi}_2$

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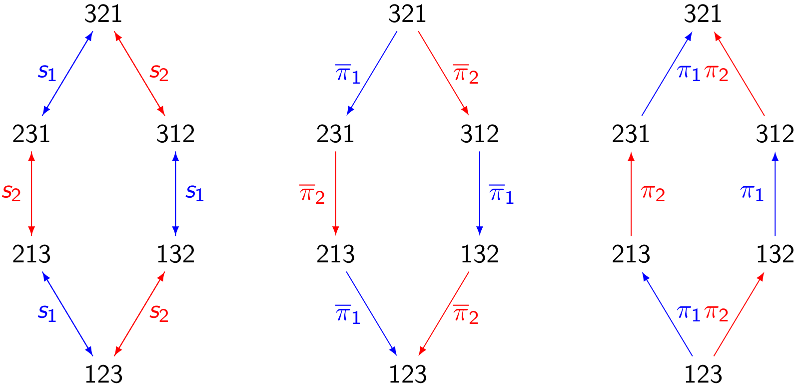

$$s_i^2 = 1 \qquad s_1 s_2 s_1 = s_2 s_1 s_2$$

Symmetric group $\mathfrak{S}_3$

$$\overline{\pi}_i^2 = \overline{\pi}_i \qquad \overline{\pi}_1\overline{\pi}_2\overline{\pi}_1 = \overline{\pi}_2\overline{\pi}_1\overline{\pi}_2$$

0-Hecke monoide $\overline{H}_0(\mathfrak{S}_3)$

$$\overline{\pi}_i^2 = \overline{\pi}_i \qquad \pi_1\pi_2\pi_1 = \pi_2\pi_1\pi_2$$

$H_0(\mathfrak{S}_3)$

Coxeter groups

Definition (Coxeter group W)

Generators : $s_1, s_2, \ldots$ (simple reflections)

Relations: $s_i^2 = 1$ and $\underbrace{s_i s_j \cdots}_{m_{i,j}} = \underbrace{s_j s_i \cdots}_{m_{i,j}}$, for $i \neq j$

Reduced words

0-Hecke monoid

Definition (0-Hecke monoid $H_0(W)$ of a Coxeter group W)

Generators : $\langle \pi_1, \pi_2, \ldots \rangle$ (simple reflections)
Relations: $\pi_i^2 = \pi_i$ and braid relations

Theorem

$|H_0(W)| = |W|$
+ *lots of nice properties (J-trivial, ...)*

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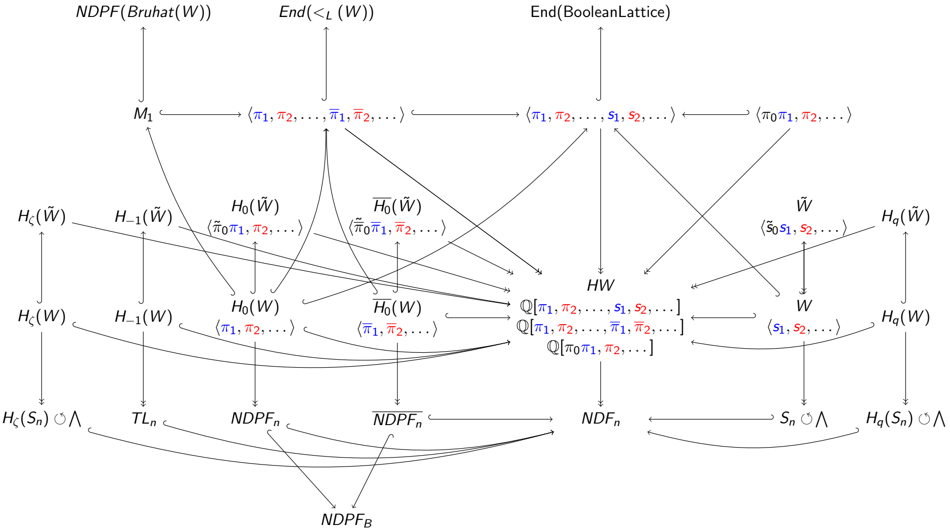
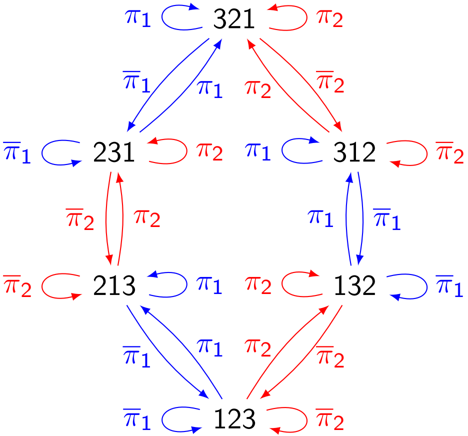
The Big Picture

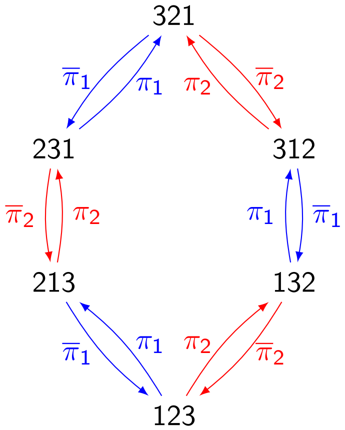


A strange cocktail: the biHecke monoid



What's the transition monoid?

A strange cocktail: the biHecke monoid



What's the transition monoid?

The biHecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots \rangle$

$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Theorem (Hivert, Schilling, T. '08)

$M(W)$ admits $|W|$ simple / indecomposable projective modules

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w . \dim P_w$$

The biHecke monoid

Question

Size of $M(W) = \langle \pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots \rangle$
$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

- How to attack such a problem?
- Generators and relations?
- Representation theory?

Theorem (Hivert, Schilling, T. '08)

$M(W)$ admits $|W|$ simple / indecomposable projective modules

- Why do we care?

$$|M(W)| = \sum_{w \in W} \dim S_w . \dim P_w$$

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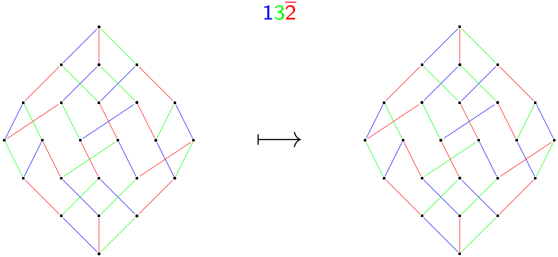

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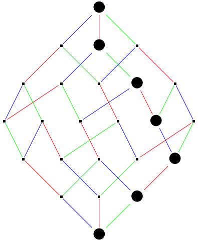
For $f \in M(W)$ *and* $w \in W$: $\quad (s_i w).f = w.f \quad$ *or* $\quad s_i\,(w.f)$

Proof.

Exchange property / associativity □

Key combinatorial lemma

$1 3 \overline{2}$



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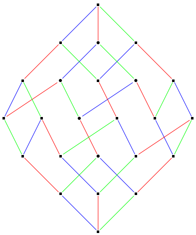

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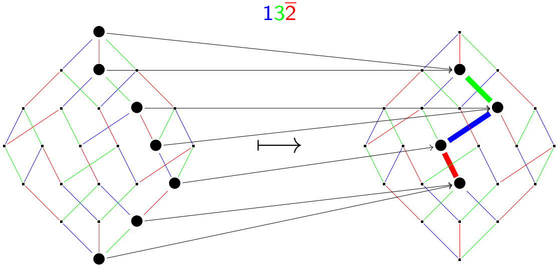

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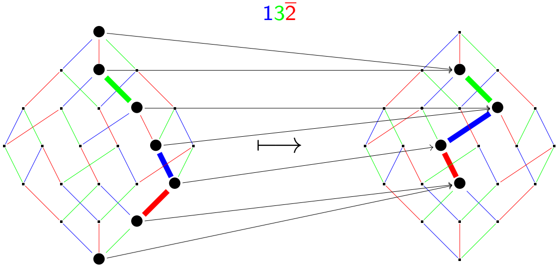

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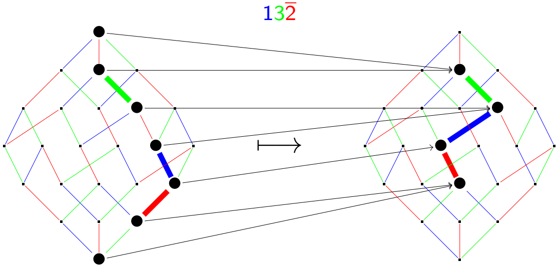

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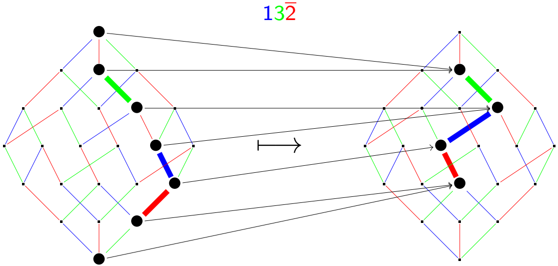

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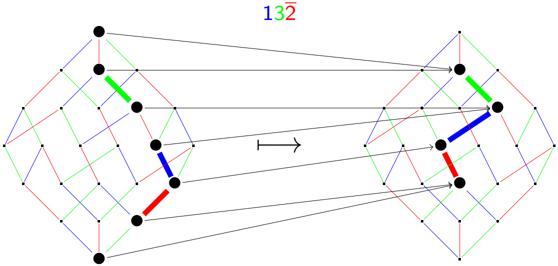

Corollary

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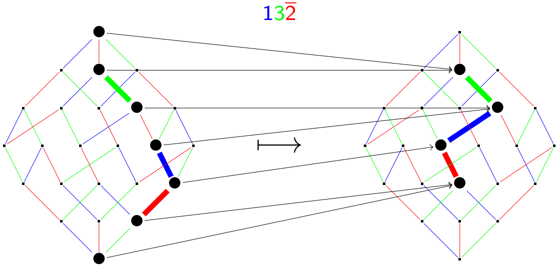

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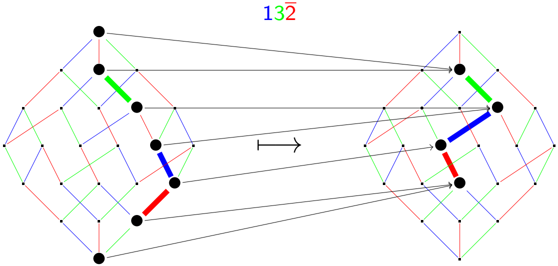

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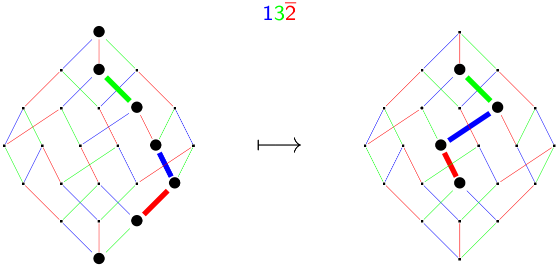

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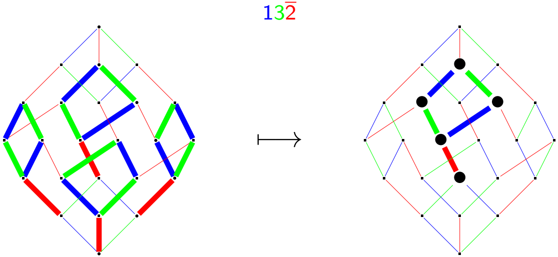

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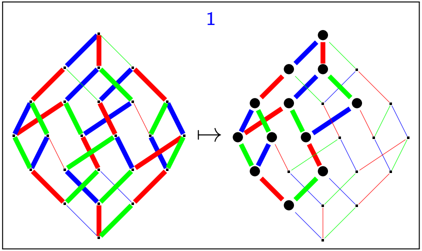
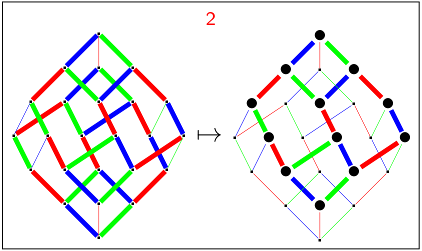
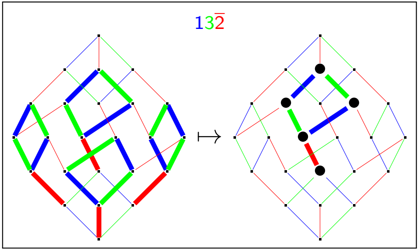
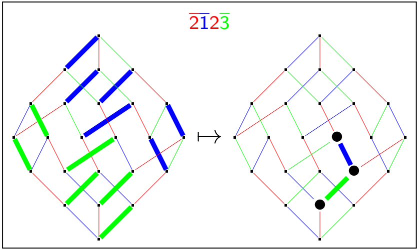
Some elements of the monoid





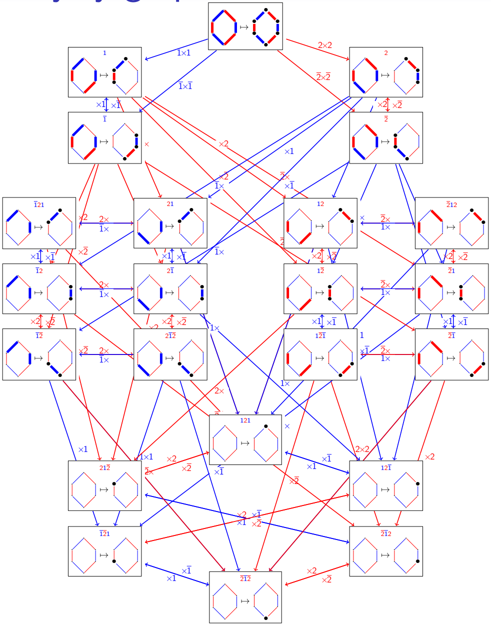




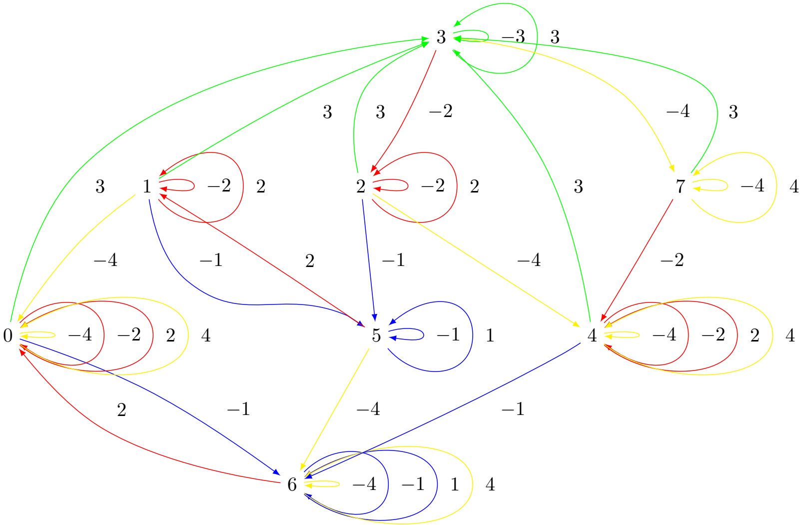
Lemma

The image set of an idempotent is an interval in left order

The left-right Cayley graph for the biHecke monoid for $\mathfrak{S}_2$



Example: a left module for the biHecke monoid for $\mathfrak{S}_5$



Linear representations

Definition (Module)

Vector space V with an action of M on V by linear operators

Equivalently: linear representation of M as submonoid of $M_n(K)$

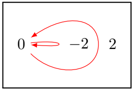
Example

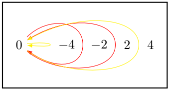
Linear module $V = \mathbb{Q}X$ associated to X

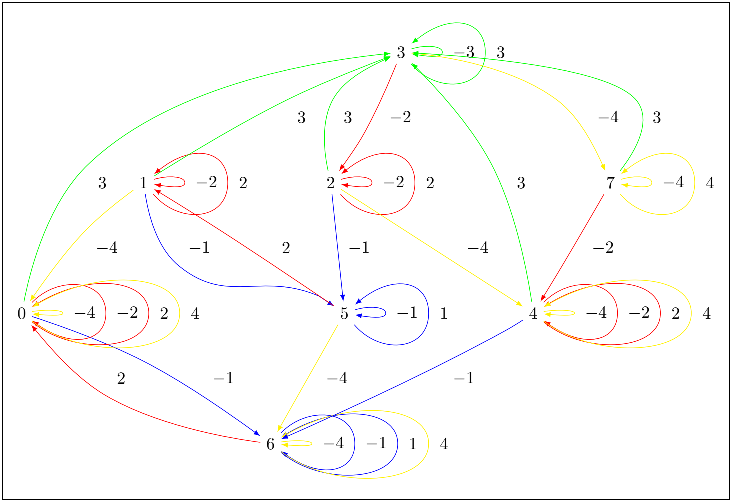
Definition

- Submodule, simple module
- Quotient module

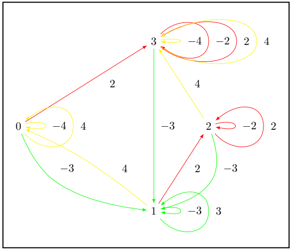
Finding submodules?

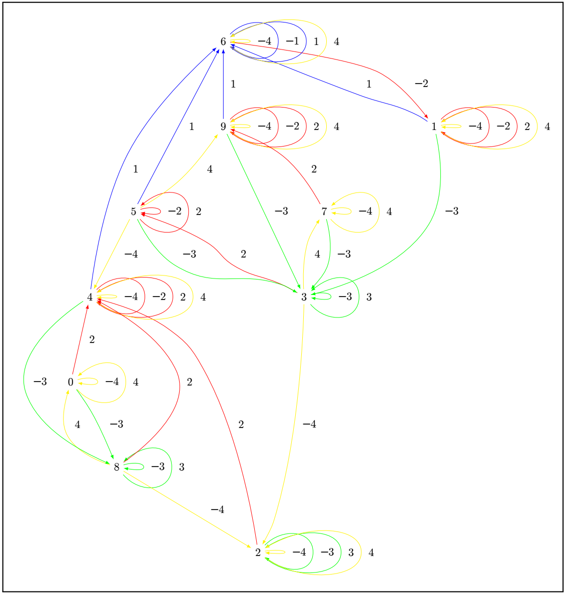






Digression: embedding submodules by linear algebra





What's known about linear representations?

Finite groups

- Semi-simple: simple = projective (characteristic 0)
- Character theory
- Fast $o(n)$ algorithms

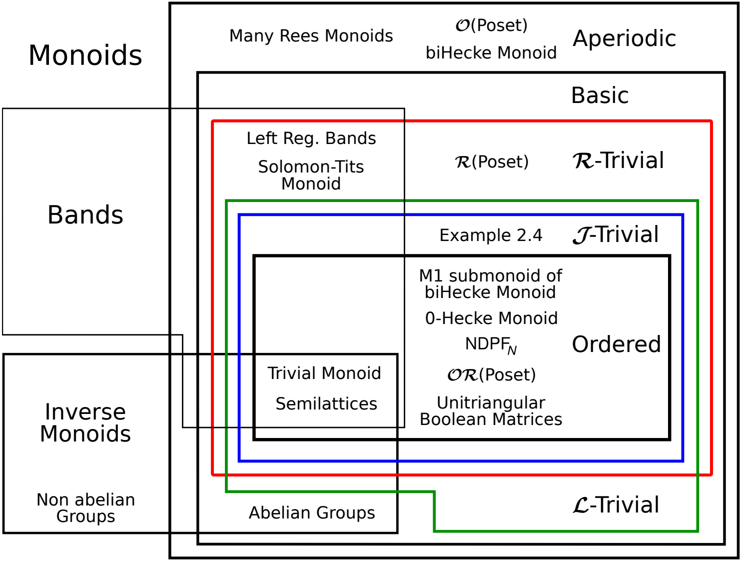
Finite dimensional algebras

- One-to-one correspondance Simple - Projective modules
- Algorithmic: minimal polynomial, linear algebra: $O(n^3)$
- In practice: dimension ≤ 1000

Monoids

- In progress (Putcha, Saliola, Steinberg, Margolis, ...)

Zoology of monoids



Goal for the rest of the talk

For an aperiodic monoid, calculate

- **Cartan matrix**
- Projective modules
- Quiver
- Radical / socle filtration

Linear refinement of the $\mathcal{R}$-preorder

Definition (Maximal composition series)

$$\{0\} = V_0 \subset \cdots \subset V_\ell = V$$

such that V_{k+1}/V_k is simple

Proposition

Composition series are not unique
The multiset $\{\{[V_{k+1}/V_k]\}\}$ of the composition factors is

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Composition factors of an $\mathcal{R}$-class modules?

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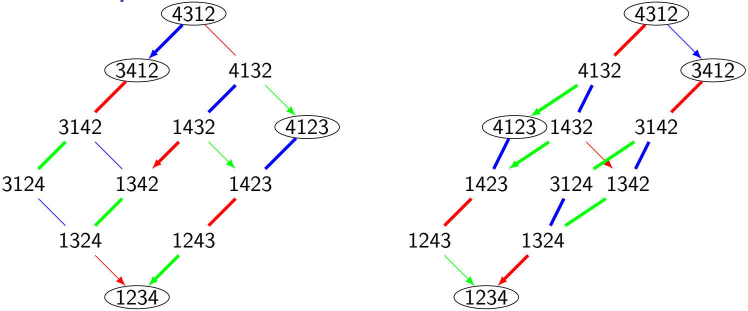

Definition (Translation algebra)

$\mathcal{H}W^{(w)} := \mathbb{Q}[\pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots]$ acting on $\mathbb{Q}.[1, w]_R$

- **Blocks**: $J = \{\}, \{1, 2\}, \{3\}, \{1, 2, 3\}$ $\implies$ Submodules P_J
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- $\mathcal{H}W^{(w)}$ quotient of $\mathbb{Q}[M(W)]$; top: simple module S_w of M
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- Generating series calculation?

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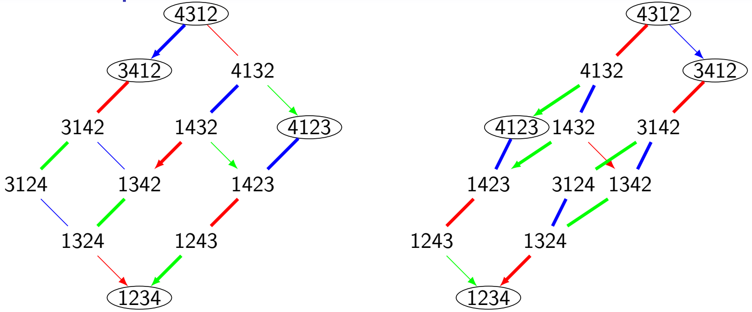

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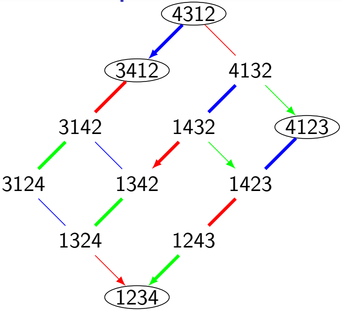

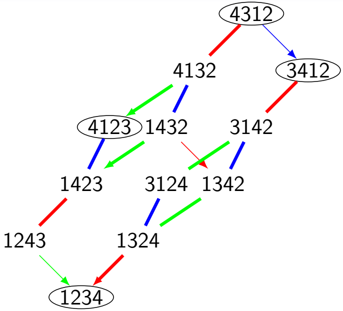

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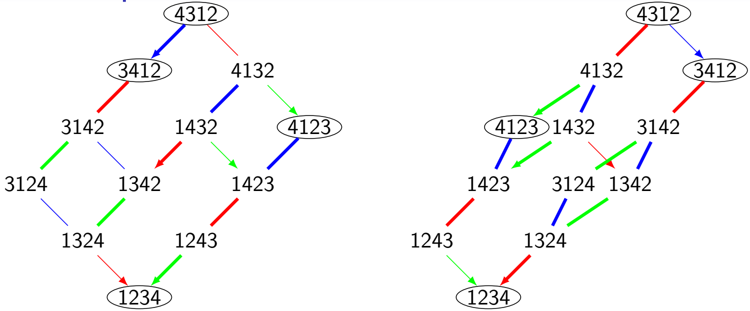

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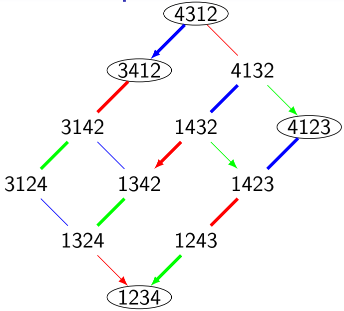

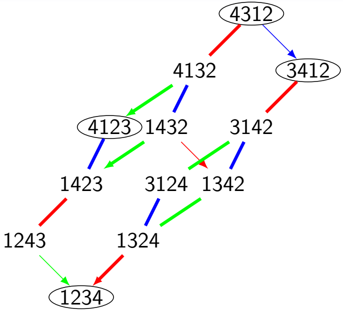

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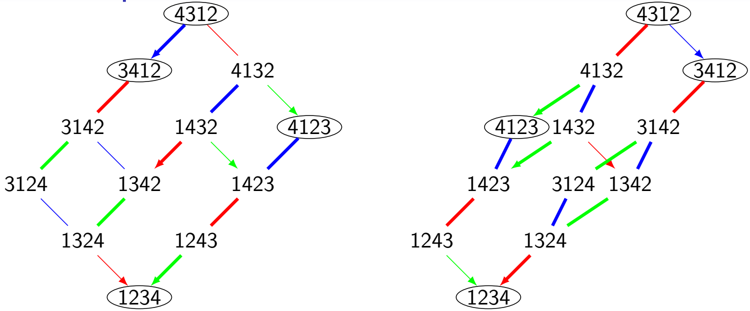

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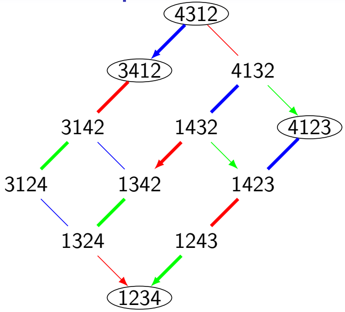

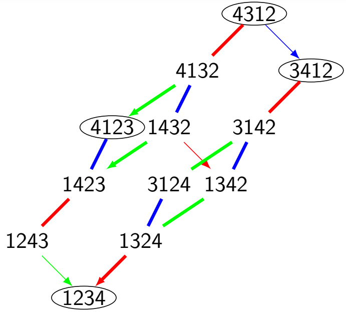

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Definition (Block of a permutation w)

- Type A: sub-permutation matrix
- Type free: J, K such that $W_J w = w W_K$

- Example: $w := 36475812$

- Simple permutation: cf. [Albert, Atkinson 05] + dim 2 posets
- $\{\texttt{blocks of } w\}$: sub-lattice of the Boolean lattice

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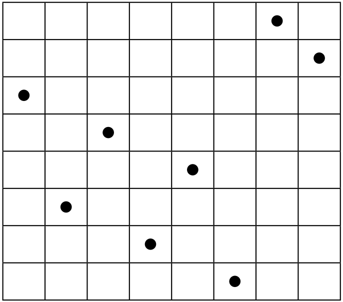

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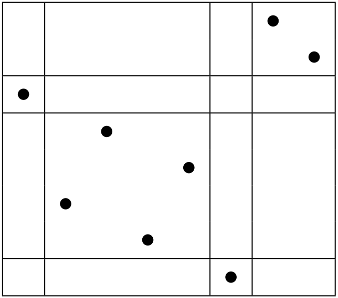

- Simple permutation: cf. [Albert, Atkinson 05] + dim 2 posets
- {blocks of w}: sub-lattice of the Boolean lattice

Blocks of permutations

Definition (Block of a permutation w)

- Type A: sub-permutation matrix
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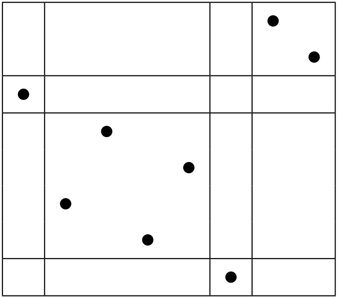

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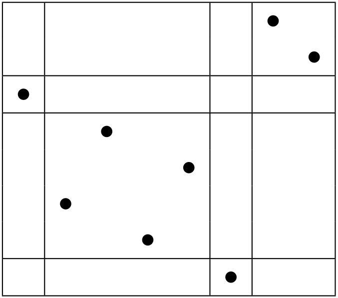

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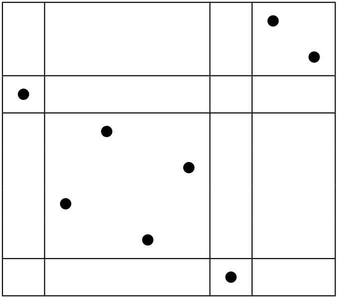

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The cutting poset

Definition (HST09: Cutting poset $(W, \sqsubseteq)$)

$u \sqsubseteq w$ if $u = w^J$ with J block

Theorem

- *Intervals are distributive lattices*
- *Meet-semi lattice*
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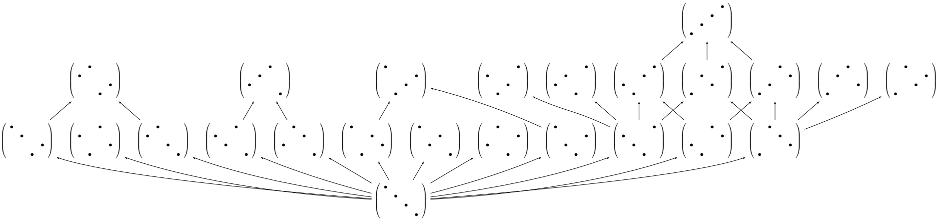

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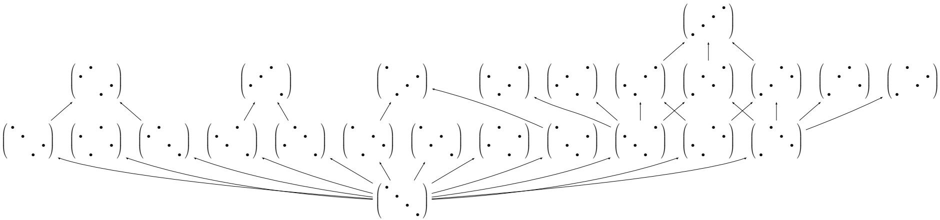

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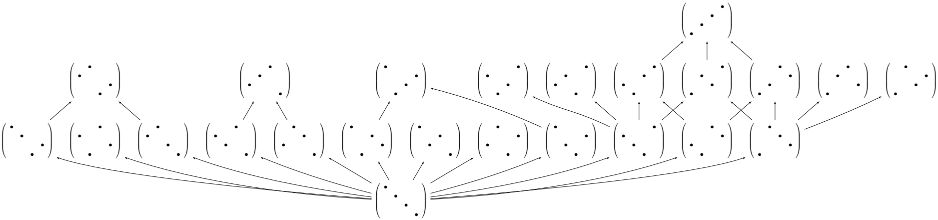

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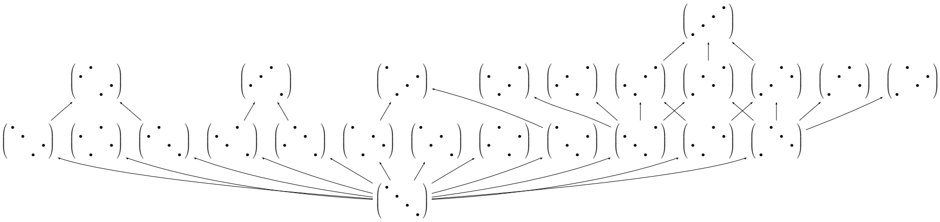

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Simple modules of an aperiodic monoid M

Proposition (Almeida, Margolis, Steinberg, Volkov, ...)

For R a regular $\mathcal{R}$-class of M,

$$\mathrm{rad}\mathbb{K}R = \{x \in \mathbb{K}R,\ x.r = 0 \forall r \in R\}$$

Equivalently: $\mathrm{rad}\mathbb{K}R$ *is the kernel of the eggbox matrix*

(boils down to aperiodic Rees matrix monoid)

Proposition (Almeida, Margolis, Steinberg, Volkov, ...)

Define $S_i := \mathbb{K}R_i/\mathrm{rad}\mathbb{K}R_i$
All simple modules of M*:* $(S_i)_{i\in I}$

Decomposing modules using embedding

Algorithm

Input:

- M: aperiodic monoid
- V: module

1. Try embedding each simple module in turn
2. Quotient out
3. Repeat

Output:

- Composition series
- Composition factors
- Socle filtration
- Using duality: radical filtration (?)

Decomposing modules using characters I

M: aperiodic monoid
I: indexing of the regular J-classes
$(e_i)_{i \in I}$: transversal of idempotents
V: right M-module

Definition

Character of an element acting on V:

$$\chi(V, m) := \mathrm{tr}_V(m)$$

Character of V:

$$\chi(V) := \sum_{i \in I} \chi(V, e_i)[C_i]$$

Alternative characteristic free definition:

Decomposing modules using characters II

Character table: $\chi(S_i)_{i \in I}$

Proposition

The character table is unitriangular w.r.t. J-order:

$$\chi(S_i) = C_i + \dots$$

Entries: non negative integers

Corollary

The character table is invertible, even over $\mathbb{Z}$

Decomposing modules using characters III

Algorithm

Input: module V

1. Precompute the character table
2. Compute $\chi(V)$
3. Compute its preimage

Output: the composition factors of V

Remarks

- *Works in all characteristic c*
- χ *characteristic free for a combinatorial module* $V = \mathbb{K}X$*:*

$$\chi(\mathbb{K}X, e_i) = |\textit{fixed points in } X|$$

Hence

- *Only the character table depends on the characteristic*

Linear refinement of $\mathcal{J}$-preorder

Definition

A: finite dimensional algebra (e. g. $A = \mathbb{Q}[M]$)
A is an A-mod-A module (or $A^{\mathrm{op}} \otimes A$-module)
Composition series: $\{0\} = A_0 \subset \cdots \subset A_\ell = A$

Proposition (Linear refinement of the eggbox picture)

$$A_{k+1}/A_k \approx_{A-\mathrm{mod}-A} L \otimes R$$

where L is a simple left module and R is a simple right module

See e. g. Curtis-Reiner

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The Cartan (invariants) matrix

Definition

$C = (c_{i,j})_{i,j}$, with:

$$c_{i,j} = |\{k,\ A_{k+1}/A_k \approx_{A-\mathrm{mod}-A} S_i \otimes S_j\}|$$

Equivalent definitions:

- On the left: $[P_j] = \sum_i c_{i,j}[S_i]$
- On the right: $[P_i] = \sum_j c_{i,j}[S_j]$
- Dimension of sandwiches by idempotents: $c_{i,j} = \dim e_i A e_j$

Cartan matrix by orthogonal idempotents

1. Build a decomposition of the identity into orthogonal idempotents e_i
2. Compute $e_i A e_j$
3. Build the projective modules as $e_i A$

Problem

Non trivial construction!

- *0-Hecke in type A: combinatorial formula [Denton'10]*
- *$\mathcal{R}$-trivial: recursive formula [Berg,Bergeron,Bhargava,Saliola'10]*
- *Aperiodic?*
- *Algebra: may require arbitrary algebraic extensions*

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Idempotent free approach?

Special case: $\mathcal{J}$-trivial monoids

Each $\mathcal{J}$-class $\{x\}$ gives a simple $A - \text{mod} - A$ module

$\Longrightarrow$ $A - \text{mod} - A$ composition series

$\text{lfix}(x)$: smallest idempotent such that $e_i x = x$

Theorem (Denton, Hivert, Schilling, T'11)

Combinatorial description of the Cartan matrix:

$$c_{i,j} = |\{x, \quad \text{lfix}(x) = i, \text{rfix}(x) = j\}|$$

Projective modules:

$$P_i = \mathbb{K}\{x, \quad \text{lfix}(x) = i\}$$

Idem for the radical and quiver.

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Radical filtration?

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Radical filtration?

Cartan matrix of aperiodic monoids using the eggbox picture

Remark

- *The composition series of $\mathbb{Q}[M]$ refines the decomposition of M into $\mathcal{J}$-classes*
- *For J a $\mathcal{J}$-class of the form $L \times R$:*

$$J \quad \approx_{\mathbb{Q}[M]-\mathrm{mod}-\mathbb{Q}[M]} \quad \mathbb{Q}L \otimes \mathbb{Q}R$$

Proposition (T.)

M_L: decomposition matrix of left class modules into simples
M_R: decomposition matrix of right class modules into simples
Then, $C = M_{\mathcal{L}}^t M_{\mathcal{R}}$

Remark: $M_{\mathcal{L}}$ and $M_{\mathcal{R}}$ are upper unitriangular

Cartan matrix of aperiodic monoids

Algorithm

Input: an aperiodic monoid

1. Construct representatives of left and right class modules
2. Construct the simple modules as quotients thereof
3. Compute the character table
4. Compute the character of each left and right class module
5. Compute the decomposition matrices $M_{\mathcal{L}}$ and $M_{\mathcal{R}}$

Output: The cartan matrix $C = M_{\mathcal{L}}^t M_{\mathcal{R}}$

Advantages

- Splits the linear algebra in small chunks
- Take advantage of the redundancy
- Rough complexity: $O(\sum_{i\in I} |R_i|^3)$
- Cartan matrix of a monoid of size 31103 in one hour

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Theoretical consequences

- Mostly characteristic free
- No algebraic extension needed (no surprise)
- Probable generalization to PIDs ($\mathbb{Z}$, ...)?
- The Cartan matrix is invertible (over $\mathbb{Z}$?)
- Various $\mathbb{Z}$-bases for the character ring
  (analog of symmetric functions?)

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- *Quiver?*
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