Alain finding a gem by computer exploration:

Factorization properties of Young's natural idempotents

Let's build one of the Young natural idempotents for the Symmetric group S_9

```
W = SymmetricGroup(9)

I = Partition([3,3,2,1])
J = Composition([2,3,4])

tI = StandardTableaux(I).last()
tJ = Tableau([[6,7,8,9], [3,4,5], [1,2]])
tIc = Tableau([[3,6,8,9],[2,5,7],[1,4]])

muI = W([1,4,2,5,7,3,6,8,9])
```

It's indexed by a pair of standard tableaux, which we show here, in French notation of course:

```
Tableaux.global_options(convention="French")
tI.pp()
print
tJ.pp()
```

```
  9
  7  8
  4  5  6
  1  2  3

  1  2
  3  4  5
  6  7  8  9
```

The idempotent is the usual product of two pieces, a sum across a row stabilizer, and an alternating sum across a column stabilizer.

```
squareI
```

$$B + B_{(7,8)} + B_{(5,6)} + B_{(5,6)(7,8)} + B_{(4,5)} + B_{(4,5)(7,8)} + B_{(4,5,6)} + B_{(4,5,6)(7,8)} + B_{(4,6,5)} + B_{(4,6,5)(7,8)} + B_{(4,6)} + B_{(4,6)(7,8)} + B_{(2,3)} + B_{(2,3)(7,8)} + B_{(2,3)(5,6)} + B_{(2,3)(5,6)(7,8)} + B_{(2,3)(4,5)} + B$$

Both pieces being large, their product is a huge linear combination of permutations. One can compute with it, but it's useless to even look at it.

```
idempotent = nablaJ * A.monomial(muI) * squareI
len(idempotent)
```

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So Alain went onto a quest for a compact representation of this object that would be amenable to scrutiny and hand manipulation.

The first step, quite natural, was to represent permutations by their Lehmer code. The second step, typical of Alain, was to encode each such code as an exponent vector. This makes the idempotent into a huge multivariate polynomial.

```
P = QQ["x1,x1,x2,x3,x4,x5,x6,x7,x8,x9"]
x = muI(P.gens())
def to_monomial(sigma):
    code = Permutation(sigma).to_lehmer_code()
    return prod( xi^ci for xi,ci in zip(x,code) )
to_polynomial = A.module_morphism(to_monomial, codomain=P)

p = to_polynomial(idempotent)
```

Here are its first 20 terms:

```
sum(p.monomials()[:20])
```

$$x_1^5x_1^5x_2^3x_3^2x_4^3x_5^2x_6x_7 + x_1^5x_1^5x_2^3x_3^2x_4^3x_5^2x_6 + x_1^5x_1^5x_2^3x_3^2x_4^3x_5^2x_7 + x_1^5x_1^5x_2^3x_3^2x_4^3x_5x_6x_7 + x_1^5x_1^5x_2^3x_3^2x_4^2x_5^2x_6x_7 + x_1^5x_1^5x_2^3x_3x_4^3x_5^2x_6x_7 + x_1^5x_1^5x_2^2x_3^2x_4^3x_5^2x_6x_7 + x_1^5x_1^4x_2^3x_3^2x_4^3x_5^2x_6x_7 + x_1^4x_1^5x_2^3x_3^2x_4^3x_5^2$$

So far, so good. But the gain is not that obvious.

Now comes the step of genius, because it is so unnatural: the multiplicative structure of the algebra of the symmetric group has nothing to do with that of polynomials. There is no reason whatsoever to believe that the multiplication of polynomials would have any *meaning*.

Yet, Alain tried to actually factor that polynomial, and here is the gem that came out:

```
factor(p)
```

evaluate

$$(x_7 - 1) \cdot (x_6 + 1) \cdot (x_4 + 1) \cdot (x_2 - 1) \cdot (x_1 + 1) \cdot (x_5^2 - x_5 + 1) \cdot (x_3^2 + x_3 + 1) \cdot (x_2^2 + 1) \cdot (x_1^2 + x_1 + 1) \cdot (-x_1^3 + x_3^2) \cdot (-x_1^4x_4^2 + x_1^2x_4^3 + x_1^4x_6 - x_4^3x_6 - x_1^2x_6^2 + x_4x_6^2)$$

Reference: *Young's natural idempotents as polynomials*, Alain Lascoux, Annals of Combinatorics 1, 1997, 91-98