Travaux Dirigés Polynomials n°1

Combinatorics and Computer Algebra Lecture

—MPRI master's—

Interpolation / Finite Differences / Newton Polynomials

The finite difference method is used, among other things, to calculate iteratively the values taken by a polynomial over the integers with no multiplication following a preliminary calculation. This method is also widely used in numerical analysis to calculate approximations of derivatives.

For $n+1$ pairs (x_i, y_i), we look for a function Φ such that $\Phi(x_i) = y_i$ for $0 \leq i \leq n$. In the case of polynomial interpolation, we look for the function Φ in the form of a polynomial. The problem is therefore reduced to finding a polynomial P of degree n such that $P(x_i) = y_i$ for $0 \leq i \leq n$.

1 Lagrange interpolation

▸ Exercice 1. (Existence and uniqueness)

Show that given $n+1$ distinct pairs $(x_0, y_0), \ldots, (x_n, y_n)$ such that $x_i \neq x_j$, there is a unique polynomial P of degree n such that $P(x_i) = y_i$ for $0 \leq i \leq n$. We assume the following formula, which gives the Vandermonde determinant :

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \ldots x_0^n \\ 1 & x_1 & x_1^2 & \ldots x_1^n \\ \vdots & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \ldots x_n^n \end{vmatrix} = \prod_{i>j} (x_i - x_j) \tag{1}$$

Application : Calculate the interpolating polynomial by replacing the coordinates of the points in the polynomial expression and solve the linear system : find the polynomial interpolating a function passing through the points $(-1, 2), (1, 4), (3, 2)$ and $(5, 1)$.

▸ Exercice 2. (Second method : Lagrange method)

The Lagrange interpolating polynomial has the form

$$\Pi(x) := \sum_{i=0}^{n} y_i L_i(x) \tag{2}$$

where the

$$L_i(x) := \prod_{j=0,\ j\neq i}^{n} \frac{x - x_j}{x_i - x_j}\ . \tag{3}$$

are called Lagrange basis polynomials.

1. Show that $L_i(x_i) = 1$ and $L_i(x_k) = 0$ if $i \neq k$.
2. Deduce that for $n+1$ distinct pairs $(x_0, y_0), \ldots, (x_n, y_n)$, the Lagrange polynomial does indeed constitute an interpolation.
3. Calculate the interpolating polynomial of the function passing through the points $(-1, 3), (0, 1), (1, 2), (3, -1)$ and $(4, -5)$.

2 Finite differences

Let $P(X)$ be a polynomial. The finite difference operator notated Δ acts on the polynomial $P(X)$ as follows :

$$\Delta P(X) := P(X+1) - P(X)\,. \tag{4}$$

▸ **Exercice 3. Some properties of the Δ operator**

1. Let $Q(X) := X^3 + 2X^2 + X + 2$. Calculate $\Delta Q(X)$.
2. Show that Δ is linear, i.e. if a and b are constants and if $P(X)$ and $Q(X)$ are polynomials, then

$$\Delta(aP(X) + bQ(X)) = a\Delta P(X) + b\Delta Q(x)\,. \tag{5}$$

3. Show that for two polynomials P and Q, we have

$$\Delta(P(X)Q(X)) = Q(X+1)\Delta P(X) + P(X)\Delta Q(X)\,. \tag{6}$$

▸ **Exercice 4. Iteration of the Δ operator**

By induction, we define $\Delta^0 P(X) := P(X)$ and $\Delta^{n+1}P(X) := \Delta(\Delta^n P(X))$ for all $n \geq 0$.

4. Using the polynomial $Q(X)$ from exercise 1, calculate $\Delta^n Q(X)$ for all $n \geq 0$. What do you notice ?
5. Show generally that if $P(X) := \sum_{i=0}^{d} a_i X^i$ is a polynomial of degree d (i.e. $a_d \neq 0$), then $\Delta P(X)$ is a polynomial of degree $d-1$. What is the leading coefficient of $\Delta P(X)$?
6. Deduce that $\Delta^d P$ is a constant polynomial equal to $d!a_d$.

▸ **Exercice 5. Calculating the values of a polynomial over the integers**

7. Let $P = aX + b$ be a polynomial of degree 1 given by $P(0)$ and $P(1)$. What is the polynomial ΔP ? Is it possible to calculate $P(2), P(3), \ldots$ without calculating $P(X)$?
8. If $P = aX^2 + bX + c$ is a polynomial of degree 2 given by $P(0)$, $P(1)$ and $P(2)$, How can we calculate $\Delta^2 P$? How can we calculate $\Delta P(2), \Delta P(3), \Delta P(4) \ldots$? How can we calculate $P(3), P(4), \ldots$?
9. Let $P(X)$ be a polynomial of degree d whose values for 0, 1, …, d are known. How can we calculate a_d with no multiplication ?
10. Provide an algorithm that calculates the values of P for $d+1, d+2, \ldots$ with no multiplication.

▸ **Exercice 6. Logic test**

Continue the following sequences

11. $U = 1, 3, 5, 7, 9, \ldots$;
12. $V = 1, 5, 11, 19, 29, \ldots$;
13. $W = 1, 9, 29, 67, 129, 221, \ldots$;
14. $X = 1, 13, 73, 241, 601, 1261, 2353, \ldots$;

▸ **Exercice 7. Newton polynomials**

The following polynomial is called a Newton polynomial of order n for any $n \leq 0$:

$$A_n(X) := \frac{X(X-1)(X-2)(X-3)\dots(X-n+1)}{n!} \tag{7}$$

15. What is the degree of $A_n(X)$?
16. Show that for $n > 0$ we have $\Delta A_n(X) = A_{n-1}(X)$.
17. Show that for any polynomial $P(X)$ of degree d there is a unique $d+1$-tuple of coefficients $(c_0, \dots c_d)$ such that

$$P(X) = \sum_{i=0}^{d} c_i A_n(X)\,. \tag{8}$$

18. How can we calculate the values of c_i efficiently ?
19. Deduce an interpolation algorithm for a function f whose values are known over the integers 0, 1, ..., n.